

A FROBENIUS-NIRENBERG THEOREM WITH PARAMETER

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ABSTRACT. The Newlander-Nirenberg theorem says that a formally integrable complex structure is locally equivalent to the complex structure in the complex Euclidean space. We will show two results about the Newlander-Nirenberg theorem with parameter. The first extends the Newlander-Nirenberg theorem to a parametric version, and its proof yields a sharp regularity result as Webster's proof for the Newlander-Nirenberg theorem. The second concerns a version of Nirenberg's complex Frobenius theorem and its proof yields a result with a mild loss of regularity.

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1. INTRODUCTION

Let D be a domain in \mathbf{R}^N and let S be a complex subbundle of $T(D) \otimes \mathbf{C}$. The rank of S over \mathbf{C} is denoted by $\text{rank}_{\mathbf{C}} S$. Assume that S is of class \mathcal{C}^r . We say that S is *formally integrable* if X, Y are \mathcal{C}^1 sections of S then the Lie bracket $[X, Y]$ remains a section of S ; it is *CR* if $S + \bar{S}$ is a complex bundle; and it is *Levi-flat* if both S and $S + \bar{S}$ are formally integrable. When S is a CR vector bundle, we call $\text{rank}_{\mathbf{C}}(S + \bar{S}) - \text{rank}_{\mathbf{C}} S$ the *CR dimension* of S . We first formulate a finite smoothness result analogous to a theorem of Nirenberg [28] for the \mathcal{C}^∞ case.

Proposition 1.1. *Let S be a Levi-flat CR vector bundle of class \mathcal{C}^r with $r \in (1, \infty)$ in a domain $D \subset \mathbf{R}^N$. For each $p \in D$ there exist a neighborhood U of p and a diffeomorphism F in $\mathcal{C}^a(U)$ for all $a < r$ such that F_*S is spanned by*

$$(1.1) \quad \partial_{\bar{z}_1}, \dots, \partial_{\bar{z}_n}, \partial_{t_1}, \dots, \partial_{t_M}.$$

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Throughout the paper, $y := (x, t, \xi)$ are the coordinates of $\mathbf{R}^N = \mathbf{R}^{2n} \times \mathbf{R}^M \times \mathbf{R}^L$, while z with $z_j = x_j + ix_{n+j}$ are the complex coordinates of \mathbf{R}^{2n} .

Let us discuss special cases of Proposition 1.1. When the S defines a real structure additionally, i.e., $S = \bar{S}$, the result is the classical Frobenius theorem [6] (see also Hawkins [14]) and it actually holds for $F \in \mathcal{C}^r$ (see Guggenheimer [12], Narasimhan [24], F. and R. Nevanlinna [26], and Proposition 4.2 below). When S defines a complex structure, i.e. its CR dimension is $\frac{N}{2}$, the result with a better regularity is the well-known Newlander-Nirenberg theorem [25]; it is sharp with $F \in \mathcal{C}^{r+1}$ when $r \in (1, \infty] \setminus \mathbf{N}$, which is a theorem of Webster [31].

We now formulate results in a parametric version. We first define Hölder spaces with parameter. Let \mathbf{N} be the set of non-negative integers and let $I = [0, 1]$. For $r \in [0, \infty)$, let $[r]$ denote the largest integer $\leq r$ and set $\{r\} = r - [r]$. When D is bounded and $0 \leq s \leq r \leq \infty$, by an element in $\mathcal{C}^{r,s}(\bar{D})$ we mean a family $\{u^\lambda; \lambda \in I\}$ of functions u^λ in \bar{D} having the property that if $i, j \in \mathbf{N}$, $i + j \leq r$ with $j \leq s$, then $\partial_y^i \partial_\lambda^j u^\lambda(y)$ are continuous in $\bar{D} \times I$ and have bounded $\mathcal{C}^{\{r\}}(\bar{D})$ -norms ($r < \infty$) as λ varies in I and bounded $\mathcal{C}^{\{s\}}(I)$ -norms ($s < \infty$) as y vary in \bar{D} . We say that a family $\{S^\lambda\}$ of CR vector bundles in D is of class $\mathcal{C}^{r,s}$, if each vector $v \in S_{p_0}^{\lambda_0}$ extends to a vector field

$$a_1^\lambda(y) \partial_{y_1} + \cdots + a_N^\lambda(y) \partial_{y_N}, \quad \{a_j^\lambda\} \in \mathcal{C}^{r,s}(\bar{D}).$$

Set $\mathcal{C}^{r-,s}(U) = \bigcap_{a < r} \mathcal{C}^{a,s}(U)$ and $\mathcal{C}^{r-,s-}(U) = \bigcap_{a < r, b < s} \mathcal{C}^{a,b}(U)$.

Let us first state a result for a family of complex structures.

Proposition 1.2. *Let $s \in \mathbf{N}$. Let D be a domain in \mathbf{R}^{2n} . Let $\{S^\lambda\}$ be a family of complex structures in D of class $\mathcal{C}^{r,s}$ with $r > 1$. For each $p \in D$ there exist a neighborhood U of p and a family $\{F^\lambda\} \in \mathcal{C}^{1,0}(U)$ of diffeomorphisms F^λ such that each $F_*^\lambda S^\lambda$ is spanned by $\partial_{\bar{z}_1}, \dots, \partial_{\bar{z}_n}$. If $r \notin \mathbf{N}$, the $\{F^\lambda\}$ is in $\mathcal{C}^{r+1,s}(U)$ when $r > s+1$. If r, s are in \mathbf{N} and $r \geq s+3$, then $\{F^\lambda\} \in \mathcal{C}^{(r+1)-,s}(U)$.*

The above result for $n = 1$, $r \in (0, \infty) \setminus \mathbf{N}$ and $s \in \mathbf{N}$ is due to Bertrand-Gong-Rosay [3]. The proof for the above result for $s \in \mathbf{N}$ and $r \in (s+1, \infty) \setminus \mathbf{N}$ is adapted from Webster [31].

The main result of this paper is the following.

Theorem 1.3. *Let D be a domain in \mathbf{R}^N with $N \geq 2$. Let $\{S^\lambda\}$ be a family of Levi-flat CR vector bundles S^λ in D of class $\mathcal{C}^{r,s}$ with $r > 1$. Assume that S^λ has CR dimension n and rank $m + n$. For each $p \in D$ there exist a neighborhood U of p and a family $\{F^\lambda\} \in \mathcal{C}^{1,0}(U)$ of diffeomorphisms F^λ so that each $F_*^\lambda S^\lambda$ is spanned by (1.1). Moreover, the $\{F^\lambda\}$ satisfies the following properties:*

- (i) $\{F^\lambda\} \in \mathcal{C}^{\infty,s}(U)$, provided $r = \infty$, and $s \in \{0\} \cup [1, \infty]$.
- (ii) $\{F^\lambda\} \in \mathcal{C}^{r-,s}(U)$, provided (a) $r > s+2 \in \mathbf{N}$, or (b) $s = 0$ and $r > 1$.
- (iii) $\{F^\lambda\} \in \mathcal{C}^{r-,s-}(U)$, provided (a) $r \geq s+3$ and $\{r\} \geq \{s\} > 0$, or (b) $r = s > 1$.

Note that Proposition 1.1 is a special case of (ii) with $s = 0$.

We will also study an *analytic* family of Frobenius structures. We will show that if $a \in (1, \infty) \setminus \mathbf{N}$, the real analytic family $\{P^\lambda\}$ of \mathcal{C}^a vector fields in \mathbf{R}^2 , defined by

$$(1.2) \quad P^\lambda = \partial_{x_1} + (|x_2|^a + \lambda) \partial_{x_2},$$

cannot be transformed into the span of ∂_{x_1} by a family $\{F^\lambda\}$ of diffeomorphisms with $\partial_\lambda F^\lambda \in \mathcal{C}^a$. In contrast, Nirenberg [28] showed that an analytic family of \mathcal{C}^∞ complex

structures can be transformed into the standard complex structure by an *analytic* family of \mathcal{C}^∞ diffeomorphisms. In [28] Nirenberg raised the question if an analytic family of \mathcal{C}^∞ complex Frobenius structures can be transformed into the standard complex Frobenius structures by a real analytic family of \mathcal{C}^∞ diffeomorphisms. The above example gives a negative answer to a finite smooth version of the analogous question; see Proposition 4.6 for the proof. However, we do not know a counter-example to Nirenberg's original question.

The loss of derivatives in an arbitrarily small Hölder exponent has also occurred in the regularity result by Gong-Webster [11] for the local CR embedding problem, which improves regularity results of Webster [31] and Ma-Michel [21]. However, it is unknown if such a loss is necessary in [11] and our results. Theorem 1.3 for the $\mathcal{C}^{\infty,\infty}$ case is due to Nirenberg [28]. See also Treves's book [30, pp. 296-297] for another proof when $s \in \mathbf{N}$ and $r = \infty$. Treves uses a method in [28] combining with the method in Webster [31]. We will also use these methods to prove Proposition 6.5, a weak form of Theorem 1.3.

We now mention several proofs of the Newlander-Nirenberg theorem. Nijenhuis and Woolf [27] reduced the required smoothness in [25]. Kohn [20] gave a proof for the \mathcal{C}^∞ case based on a solution to the $\bar{\partial}$ -Neumann problem. Hörmander [17] studied the L^2 regularity of solutions of first-order differential operators defined by the complex Frobenius structures that satisfy a convexity condition. Malgrange [22] proved the Newlander-Nirenberg theorem by a reduction to partial differential equations with real analytic coefficients, for which the quasi-linear elliptic theory applies. Newlander-Nirenberg [25] and Nijenhuis-Woolf [27] proved a parametric version of Newlander-Nirenberg theorem. Hill-Taylor [15, 16] proved the Newlander-Nirenberg type theorem and complex Frobenius structures of less than \mathcal{C}^1 smoothness. We note that Nirenberg's complex Frobenius theorem has an application in Kodaira-Spencer [19]. Furthermore, the vector fields (1.1) define the complex differential $\mathcal{D} := \bar{\partial} + d_t$ in $\mathbf{C}^n \times \mathbf{R}^M$. Hanges and Jacobowitz [13] prove the interior \mathcal{C}^∞ regularity of \mathcal{D} equations for smoothly bounded domains in $\mathbf{C}^n \times \mathbf{R}^M$.

Our proof is motivated by Webster's methods [31, 32] of using homotopy formulae. We will also need improved iteration methods in Gong-Webster [9, 11]. When dealing with a family of complex structures, we will use the Koppelman-Leray homotopy formula as in [31]. For the general complex Frobenius structures, we need a homotopy formula due to Treves [30], which is a combination of the Poincaré lemma and the Bochner-Martinelli and Koppelman-Leray formulae.

The paper is organized as follows. In section 2, we describe Hölder spaces and give an initial normalization for the complex Frobenius structures. In section 3, we describe some sharp interpolation inequalities in Hölder norms. For the exposition purpose, the proofs of these inequalities are given in the appendix. In section 4 we adapt a proof of the Frobenius theorem in F. and R. Nevanlinna [26] for a parametric version and we provide examples to discuss the exact regularity of Theorem 1.3. In section 5, we recall the homotopy formula for \mathcal{D} by Treves [30] and adapt Webster's interior estimates [31] for the homotopy operators for forms depending on a parameter. In section 6 we prove Proposition 1.2. In section 7, we derive a differential complex arising from a complex Frobenius structure and apply Treves's homotopy formula to describe a general rapid iteration procedure to be used in section 8. In section 8, using a Nash-Moser smoothing technique, we prove the main result.

2. HÖLDER SPACES AND AN INITIAL NORMALIZATION

In this section, we discuss properties of the Hölder spaces mentioned in the introduction. The reader can find applications of these spaces in Nijenhuis and Woolf [27], Bertrand-Gong [2], and Gong-Kim [8]. We also obtain an initial normalization of complex Frobenius structures at a given point p .

Recall that $z_\alpha = x_\alpha + ix_{n+\alpha}$, and that $y = (\operatorname{Re} z, \operatorname{Im} z, t, \xi)$ are coordinates of $\mathbf{R}^N = \mathbf{R}^{2n} \times \mathbf{R}^M \times \mathbf{R}^L$. Let k, j be non negative integers and let $\alpha, \beta \in [0, 1]$. Let D be a bounded domain in \mathbf{R}^N and I be a finite and closed interval in \mathbf{R} . The $\mathcal{C}^{k+\alpha}(\overline{D})$ is the standard Hölder space with norm $|\cdot|_{\overline{D}; k+\alpha}$. For a family $u := \{u^\lambda\}$ of functions u^λ in \overline{D} , by $u \in \mathcal{C}_*^{k+\alpha, j+\beta}(\overline{D})$ we mean that $\partial_y^m \partial_\lambda^i u^\lambda(y)$ are continuous in $\overline{D} \times I$ and have bounded \mathcal{C}^α norms on \overline{D} as λ varies in I and bounded \mathcal{C}^β norms on I as y vary in \overline{D} , for all $m \leq k$ and $i \leq j$. We will take $I = [0, 1]$ unless specified otherwise. Throughout the paper ∂^k denotes the set of partial derivatives of order k in $y \in \mathbf{R}^N$, and \mathbf{B}_ρ^M denotes the ball in \mathbf{R}^M of radius ρ centered at the origin.

The norm on $\mathcal{C}_*^{k+\alpha, j+\beta}(\overline{D})$ is defined by

$$|u|_{D; k+\alpha, j+\beta} = \max_{0 \leq i \leq j, \ell \leq k} \left\{ \sup_{\lambda \in I} |\{\partial^i \partial_\lambda^\ell u^\lambda\}|_{\overline{D}; \alpha}, \sup_{y \in \overline{D}} |\{\partial_y^i \partial_\bullet^\ell u^\bullet\}|_{I; \beta} \right\}.$$

Assuming further that $k \geq j$, we define

$$\mathcal{C}^{k+\alpha, j+\beta}(\overline{D}) = \bigcap_{i=0}^j \mathcal{C}_*^{k-i+\alpha, i+\beta}(\overline{D}),$$

$$\|u\|_{D; k+\alpha, j+\beta} = \max\{|u|_{D; k-i+\alpha, i+\beta} : 0 \leq i \leq j\}.$$

When the domain D is clear in the context, we abbreviate $|\cdot|_{D; r}$, $|\cdot|_{D; r, s}$, $|\cdot|_{D; r, s}$, $\|\cdot\|_{D; r, s}$ by $|\cdot|_r$, $|\cdot|_{r, s}$, and $\|\cdot\|_{r, s}$, respectively. Note that the spaces $\mathcal{C}_*^{a, b}$, $\mathcal{C}^{r, s}$ are the same as $\mathcal{B}_*^{a, b}$, $\mathcal{B}^{r, s}$ in [2].

To simplify notation, let $\partial_{\overline{\alpha}} = \partial_{\overline{z}_\alpha}$, $\partial_\alpha = \partial_{z_\alpha}$, $\partial'_m = \partial_{t_m}$, and $\partial''_\ell = \partial_{\xi_\ell}$. Throughout the paper, repeated indices are summed over.

Lemma 2.1. *Let $r \geq \max\{1, s\}$. Let $\{S^\lambda\} \in \mathcal{C}^{r, s}(\overline{D})$ be a family of CR vector bundles in D with constant rank $M+n$ and CR dimension n . For each $p \in D$, there exists a family of affine transformations φ^λ of \mathbf{R}^N , sending p to the origin, so that the coefficients of φ^λ are in $\mathcal{C}^s(I)$ in λ , while the fiber at the origin of each $\varphi_*^\lambda S$ is spanned by $\partial'_m, \partial_{\overline{\alpha}}$ with $1 \leq m \leq M$, $1 \leq \alpha \leq n$. Consequently, near the origin of \mathbf{R}^N , $\varphi_*^\lambda S^\lambda$ has a unique basis*

$$(2.1) \quad X_m^\lambda = \partial'_m + \operatorname{Re}(B_{m\beta}^\lambda \partial_\beta) + b_{m\ell}^\lambda \partial''_\ell, \quad 1 \leq m \leq M,$$

$$(2.2) \quad Z_\alpha^\lambda = \partial_{\overline{\alpha}} + A_{\overline{\alpha}\beta}^\lambda \partial_\beta + a_{\overline{\alpha}\ell}^\lambda \partial''_\ell, \quad 1 \leq \alpha \leq n,$$

where $a_{\overline{\alpha}\ell}^\lambda, A_{\overline{\alpha}\beta}^\lambda, B_{m\beta}^\lambda$ are complex-valued, $b_{m\ell}^\lambda$ are real-valued, and they vanish at the origin.

Proof. It is convenient to work with K^λ , the set of complex-valued 1-forms in \mathbf{R}^N that annihilate S^λ . We are normalizing the linear part of S^λ at a point p . We may assume that $p = 0$ and for simplicity K^λ, S^λ denote the fibers of K^λ, S^λ at the origin.

In the following, all basis are over \mathbf{C} . Note that $L = \dim_{\mathbf{C}} K^\lambda \cap \overline{K^\lambda}$. For λ near a given value, $K^\lambda \cap \overline{K^\lambda}$ has a basis of real 1-forms $\eta_1^\lambda, \dots, \eta_L^\lambda$ of which the coefficients are \mathcal{C}^s functions in λ . We can also find ω_α^λ , $1 \leq \alpha \leq n$, so that $\{\omega_\alpha^\lambda, \eta_\ell^\lambda\}$ is a basis of K^λ . Then $\{\omega_\alpha^\lambda, \eta_\ell^\lambda, \overline{\omega}_\alpha^\lambda\}$ is a basis of $K^\lambda + \overline{K^\lambda}$. Finally we complete them with real forms $\theta_1^\lambda, \dots, \theta_M^\lambda$ so

that $\{\omega_\alpha^\lambda, \eta_\ell^\lambda, \overline{\omega}_\alpha^\lambda, \theta_m^\lambda\}$ is a basis of $T_0^*\mathbf{R}^N \otimes \mathbf{C}$. All coefficients of the forms are of class C^s for λ near a given value.

We cover I by intervals I_1, \dots, I_k which are either open or of the forms $[0, a), (b, 1]$ so that I_j contains $[\lambda_{j-1}, \lambda_j]$, with $\lambda_0 = 0$ and $\lambda_k = 1$. Furthermore, in I_j there are bases

$$\{\eta_{j\ell}^\lambda\}, \quad \{\eta_{j\ell}^\lambda, \omega_{j\alpha}^\lambda\}, \quad \{\eta_{j\ell}^\lambda, \omega_{j\alpha}^\lambda, \overline{\omega_{j\alpha}^\lambda}, \theta_{jm}^\lambda\}$$

for $K^\lambda \cap \overline{K}^\lambda$, K^λ , and $T_0^*\mathbf{R}^N \otimes \mathbf{C}$, respectively. We also assume the coefficients of these forms are in $C^s(I_j)$. Next, we match the bases at the end points of $[\lambda_{j-1}, \lambda_j]$. At $\lambda = \lambda_1$, we have

$$\begin{aligned} \eta_{1\ell}^{\lambda_1} &= A_{\ell\ell'} \eta_{2\ell'}^{\lambda_1}, & \omega_{1\alpha}^{\lambda_1} &= B_{\alpha\beta} \omega_{2\beta}^{\lambda_1} + b_{\alpha\ell} \eta_{2\ell}^{\lambda_1}, \\ \theta_{1m}^{\lambda_1} &= c_{mm'} \theta_{2m'}^{\lambda_1} + \text{Re}\{C_{m\alpha} \omega_{2\alpha}^{\lambda_1}\} + d_{m\ell} \eta_{2\ell}^{\lambda_1}. \end{aligned}$$

Let us replace $\theta_{2m}^\lambda, \omega_{2\alpha}^\lambda, \eta_{2\ell}^\lambda$ by the above linear combinations in which λ_1 is replaced by the variable $\lambda \in I_2$. We repeat the procedure for I_3, \dots, I_p successively. Thus we may assume that at the origin of \mathbf{R}^N

$$\eta_{j\ell}^{\lambda_j} = \eta_{(j+1)\ell}^{\lambda_j}, \quad \omega_{j\alpha}^{\lambda_j} = \omega_{(j+1)\alpha}^{\lambda_j}, \quad \theta_{jm}^{\lambda_j} = \theta_{(j+1)m}^{\lambda_j}.$$

We have obtained continuous bases that are piecewise smooth in class C^s . Using a partition of unity for I , we find bases $\{\eta_\ell^\lambda\}$, $\{\eta_\ell^\lambda, \omega_\alpha^\lambda\}$, $\{\eta_\ell^\lambda, \omega_\alpha^\lambda, \overline{\omega_\alpha^\lambda}, \theta_m^\lambda\}$ of $(S^\lambda)^* \cap \overline{(S^\lambda)^*}$, $(S^\lambda)^*$, and $T_0^*\mathbf{R}^N \otimes \mathbf{C}$, respectively. Moreover, the coefficients of the bases are in $C^s(I)$, and $\theta_m^\lambda, \eta_\ell^\lambda$ remain real-valued.

We express uniquely

$$\omega_\alpha^\lambda = df_\alpha^\lambda, \quad \theta_m^\lambda = dg_m^\lambda, \quad \eta_\ell^\lambda = dh_\ell^\lambda,$$

where $f_\alpha^\lambda, g_m^\lambda, h_\ell^\lambda$ are linear functions in $(\text{Re } z, \text{Im } z, t, \xi)$. Let

$$\varphi^\lambda: \hat{z}_\alpha = f_\alpha^\lambda(z, t, \xi), \quad \hat{t}_m = g_m^\lambda(z, t, \xi), \quad \hat{\xi}_\ell = h_\ell^\lambda(z, t, \xi).$$

Then $\{\varphi^\lambda\}$ is $C^{\infty, s}$ near the origin, and at the origin the pull-backs of $d\hat{z}_\alpha, d\hat{t}_m$ via φ^λ are ω_α^λ and θ_m^λ , respectively. Using the new linear coordinates, we have found, with possible new linear combination, bases

$$\begin{aligned} \omega_\alpha^\lambda &= \tilde{A}_{\alpha\beta}^\lambda dz_\beta + \tilde{B}_{\alpha\beta}^\lambda d\bar{z}_\beta + \tilde{C}_{\alpha m}^\lambda dt_m + \tilde{D}_{\alpha\ell}^\lambda d\xi_\ell, \\ \eta_\ell^\lambda &= \text{Re}\{\tilde{a}_{\ell\alpha}^\lambda dz_\alpha\} + \tilde{c}_{\ell m}^\lambda dt_m + \tilde{d}_{\ell\ell'}^\lambda d\xi_{\ell'}. \end{aligned}$$

At the origin, $(\tilde{A}_{\alpha\beta}^\lambda)$ and $(\tilde{d}_{\ell\ell'}^\lambda)$ are identity matrices and all other coefficients vanish. Near the origin, the vector fields annihilated by the forms have a unique basis of the form (2.1)-(2.2). The initial normalization is achieved. \square

Following Webster [31], we call $\{X_m^\lambda, Z_\alpha^\lambda\}$ the *adapted frame* of S^λ . The adapted frame will be useful to reformulate the integrability via differential forms for the $d_t + \bar{\partial}$ complex.

Another consequence of the proof of Lemma 2.1 is to identify suitable homogeneous solutions for $\{S^\lambda\}$ with the normalization of $\{S^\lambda\}$.

Proposition 2.2. *Let D be a domain in \mathbf{R}^N and let $p \in D$. Let $\{S^\lambda\}$ be a family of CR vector bundles in D of class $C^{r,s}$ with $r \geq \max\{1, s\}$. Suppose that S^λ have rank $n + M$ and CR dimension n . Let $L = N - 2n - M$. Suppose that for each $\lambda_0 \in I$ there are an open interval J with $\lambda_0 \in J$ or $\lambda_0 \in \bar{J}$ for $\lambda_0 = 0, 1$, a neighborhood U of p , complex-valued functions $f^\lambda = (f_1^\lambda, \dots, f_n^\lambda)$, and real functions $h^\lambda = (h_1^\lambda, \dots, h_L^\lambda)$ for $\lambda \in \bar{J}$ so that $\{(f^\lambda, h^\lambda); \lambda \in J\}$ are in $C^{r,s}(U)$, and f^λ, h^λ are annihilated by sections of S^λ . Assume further*

that the Jacobian matrix of $(f^\lambda, \overline{f^\lambda}, h^\lambda)$ has rank $2n+L$ at p . Then there exist a neighborhood V of p and $\{\varphi^\lambda; \lambda \in I\} \in \mathcal{C}^{r,s}(\overline{V})$ such that each φ^λ is a diffeomorphism defined in V and $\varphi_*^\lambda(S^\lambda)$ is spanned by

$$\partial_{\overline{1}}, \dots, \partial_{\overline{n}}, \quad \partial'_1, \dots, \partial'_M.$$

Proof. Let ω_α^λ and η_ℓ^λ be the linear forms such that at the origin of \mathbf{R}^N , $\omega_\alpha^\lambda = df_\alpha^\lambda$ and $\eta_\ell^\lambda = dh_\ell^\lambda$. By an argument similar to the proof of Lemma 2.1, we can find linear real 1-forms $\theta_1^\lambda, \dots, \theta_M^\lambda$ with coefficients in $C^s([0, 1])$ so that $\omega_\alpha^\lambda, \overline{\omega_\alpha^\lambda}, \theta_m^\lambda, \eta_\ell^\lambda$ form a basis of $T_0^* \mathbf{R}^N \otimes \mathbf{C}$. Note that when $s \leq r < s+1$, the coefficients of ω_α^λ and η_ℓ^λ are \mathcal{C}^{r-1} in λ . We first find θ_m^λ of which the coefficients are of \mathcal{C}^{r-1} in λ . Then by an approximation, we can replace them by forms of which the coefficients are \mathcal{C}^s in λ . Since θ_m^λ is linear, then $\theta_m^\lambda = dg_m^\lambda$ for a linear function g_m^λ . Define

$$\varphi^\lambda: z = f^\lambda(\hat{z}, \hat{t}, \hat{\xi}), \quad t = g^\lambda(\hat{z}, \hat{t}, \hat{\xi}), \quad \xi = h^\lambda(\hat{z}, \hat{t}, \hat{\xi}).$$

Then $\varphi_*^\lambda S^\lambda$ is spanned by $\partial_{\overline{\alpha}}, \partial'_m$. □

In the definition of $\mathcal{C}^{r,s}$ spaces, Lemma 2.1, Proposition 2.2 and some propositions in the paper, it suffices to require that $r \geq \max\{1, s\}$ without the restriction of $\{r\} \geq \{s\}$. In this paper, the use of the chain rule is indispensable and thus it restricts the exponents r, s as shown by the following.

Remark 2.3. Let U, V be bounded domains in Euclidean spaces. Suppose that there is a constant $C > 0$ so that two points p, q in \overline{U} can be connected by a piecewise smooth curve of length at most $C|p - q|$. Suppose that G^λ map \overline{U} into \overline{V} . If $\{f^\lambda\} \in \mathcal{C}^{r,s}(\overline{V})$ and $\{G^\lambda\} \in \mathcal{C}^{r,s}(\overline{U})$ then $\{f^\lambda \circ G^\lambda\} \in \mathcal{C}^{r,s}(\overline{U})$, provided

$$(2.3) \quad s = 0; \text{ or } 1 \leq s \leq r \text{ and } \{s\} \leq \{r\}.$$

Assume that r and s satisfy (2.3). Let $\{S^\lambda\} \in \mathcal{C}^{r,s}$ be as in Lemma 2.1. Then the adapted frame $\{X_m^\lambda, Z_\alpha^\lambda\}$ are in $\mathcal{C}^{r,s}$. Since the coefficients $A^\lambda, B^\lambda, a^\lambda, b^\lambda$ in the frame vanish at the origin, by dilation $(z, t, \xi) \rightarrow \delta(z, t, \xi)$ we achieve

$$(2.4) \quad \|\{(A^\lambda, B^\lambda, a^\lambda, b^\lambda)\}\|_{B_{1;r,s}} \leq \epsilon$$

for a given $\epsilon > 0$.

Throughout the rest of paper, we assume that r, s satisfy (2.3).

3. PRELIMINARY AND INTERPOLATION INEQUALITIES

The main purpose of this section is to acquaint the reader with some interpolation inequalities for Hölder norms on domains with the cone property; see the appendix for the definition of the cone property. The reader is referred to Hörmander [18] and Gong-Webster [10] for the proof of these inequalities. The parametric version of those inequalities are proved in appendix A.

If D, D_1, D_2 are domains of the cone property, we have

$$\begin{aligned} |u|_{D; (1-\theta)a+\theta b} &\leq C_{a,b} |u|_{D;a}^{1-\theta} |u|_{D;b}^\theta, \\ |f_1|_{a_1+b_1} |f_2|_{a_2+b_2} &\leq C_{a,b} (|f_1|_{a_1+b_1+b_2} |f_2|_{a_2} + |f_1|_{a_1} |f_2|_{a_2+b_1+b_2}). \end{aligned}$$

See [18, Theorem A.5] and [10, Proposition A.4]. Here $|f_i|_{a_i} = |f_i|_{D_i; a_i}$. Let D_i be finitely many domains of the cone property. If $|u_i|_{a_i} = |u_i|_{D_i; a_i}$, then

$$\prod_{j=1}^m |u_j|_{b_j+a_j} \leq C_{a,b}^m \sum_{j=1}^m |u_j|_{b_j+a_1+\dots+a_m} \prod_{i \neq j} |u_i|_{b_i}.$$

To deal with two Hölder exponents in y, λ variables, we introduce the following notation

$$\begin{aligned} |f|_{D;a,0} \odot |g|_{D';0,b} &:= |f|_{D;a,0} |g|_{D';0,[b]} + |f|_{D;[a],0} |g|_{D';0,b}, \\ Q_{D,D';a,b}^*(f, g) &:= |f|_{D;a,0} \odot |g|_{D';0,b} + |f|_{D;0,b} \odot |g|_{D';a,0}, \\ |g|_{D';0,b} \odot |f|_{D;a,0} &= |f|_{D;a,0} \odot |g|_{D';0,b} \\ (3.1) \quad Q_{D,D';r,s}(f, g) &:= \sum_{0 \leq k \leq [s]} Q_{D,D';r-j,j+\{s\}}^*(f, g), \quad [r] \geq [s], \\ \hat{Q}_{D,D';r,s}(f, g) &:= \|f\|_{D;r,s} |g|_{D';0,0} + |f|_{D;0,0} \|g\|_{D';r,s} + Q_{D,D';r,s}(f, g). \end{aligned}$$

We will write $Q_{D,D';a,b}^*$ as $Q_{a,b}^*$, when the domains are clear from the context, and the same abbreviation applies to Q and \hat{Q} . From Lemma A.1, we have

Lemma 3.1. *For each i , let D_i be a domain in \mathbf{R}^{n_i} with the cone property and let a_i, b_i, r_i, s_i be non-negative real numbers. Assume that $(r_1, \dots, r_m, a_1, \dots, a_m)$ or $(s_1, \dots, s_m, b_1, \dots, b_m)$ is in \mathbf{N}^{2m} , or at most one of $(r_1, s_1), \dots, (r_m, s_m)$ is not in \mathbf{N}^2 . Then*

$$\begin{aligned} \prod_{i=1}^m |f_i|_{D_i; r_i, s_i} &\leq C_{r,s}^m \left\{ \sum_i |f_i|_{r,s} \prod_{\ell \neq i} |f_\ell|_{0,0} + \sum_{i \neq j} Q_{r,s}^*(f_i, f_j) \prod_{\ell \neq i,j} |f_\ell|_{0,0} \right\}, \\ \prod_{i=1}^m |f_i|_{D_i; r_i+a_i, s_i+b_i} &\leq C_{r,s,a,b}^m \left\{ \sum_i |f_i|_{r_i+a_i, s_i+b_i} \prod_{\ell \neq i} |f_\ell|_{r_\ell, s_\ell} \right. \\ &\quad \left. + \sum_{i \neq j} |f_i|_{r_i+a_i, s_i} |f_j|_{r_j, s_j+b_j} \prod_{\ell \neq i,j} |f_\ell|_{r_\ell, s_\ell} \right\}. \end{aligned}$$

Here $a = \sum a_i$, $b = \sum b_i$, etc.. Assume further that $r_i \geq s_i$ for all i . Then

$$\|f_1\|_{D_1; r_1, s_1} \cdots \|f_m\|_{D_m; r_m, s_m} \leq C_{r,s}^m \sum_{i \neq j} \hat{Q}_{r,s}(f_i, f_j) \prod_{\ell \neq i,j} |f_\ell|_{0,0}.$$

Let $D_\rho = B_\rho^{2n} \times B_\rho^M \times B_\rho^L$, and set $\|\cdot\|_{\rho; r, s} = \|\cdot\|_{D_\rho; r, s}$. Throughout the paper $s_* = 0$ for $s = 0$, and $s_* = 1$ for $s \geq 1$. By Proposition A.10, we have

Proposition 3.2. *Let $0 < \rho < \infty$, $0 < \theta < 1/4$ and $\rho_j = (1 - \theta)^j \rho$. Let $F^\lambda = I + f^\lambda$ map D_ρ into \mathbf{R}^N . Assume that $f \in \mathcal{C}^{1,0}(D_\rho)$ and*

$$f^\lambda(0) = 0, \quad \{\partial f^\lambda\}|_{\rho; 0,0} \leq \theta/C_N.$$

Then $F^\lambda: D_\rho \rightarrow D_{(1-\theta)^{-1}\rho}$ are injective. There are $G^\lambda = I + g^\lambda$ satisfying $g^\lambda(0) = 0$ and

$$\begin{aligned} G^\lambda: D_{\rho_1} &\rightarrow D_\rho, \quad F^\lambda \circ G^\lambda = I \quad \text{on } D_{\rho_1}, \\ G^\lambda \circ F^\lambda &= I \quad \text{on } D_{\rho_2}. \end{aligned}$$

Assume further that r, s satisfy (2.3), $1/4 < \rho < 2$, and $|f|_{\rho;1,s_*} \leq 1$. Then

$$(3.2) \quad \begin{aligned} & \|g\|_{\rho_1;r,s} \leq C_r \{ \|f\|_{\rho;r,s} + \|f\|_{\rho;s+1,s} \odot \|f\|_{\rho;r,s_*} \}, \\ & \| \{ u^\lambda \circ (F^\lambda)^{-1} \} \|_{\rho_1;r,s} \leq C_r \{ |u|_{\rho;1,s_*} \|f\|_{\rho;r,s} + \|u\|_{\rho;s+1,s} \odot \|f\|_{\rho;r,s_*} \\ & \quad + \|u\|_{\rho;r,s} + |u|_{\rho;1,0} \|f\|_{\rho;s+1,s} \odot \|f\|_{\rho;r,s_*} + \|u\|_{\rho;r,s_*} \odot \|f\|_{\rho;s+1,s} \}. \end{aligned}$$

By Proposition A.11, we have the following.

Proposition 3.3. *Let $F_i^\lambda = I + f_i^\lambda: D_i \rightarrow D_{i+1}$. Assume that D_1, \dots, D_{m+1} have the cone property of which $C_*(D_i)$ are independent of i (see Appendix A for notation) and*

$$f_i^\lambda(0) = 0, \quad |f_i|_{D_i;1,s_*} \leq 1.$$

Let $\|f_i\|_{r,s} = \|f_i\|_{D_i;r,s}$. Suppose that r, s satisfy (2.3). Then

$$\begin{aligned} & \| \{ u^\lambda \circ F_m^\lambda \circ \dots \circ F_1^\lambda \} \|_{D_1;r,s} \leq C_r^m \left\{ \|u\|_{D_{m+1};r,s} + |u|_{D_{m+1};1,0} \sum_{i \leq j} \|f_i\|_{s+1,s} \odot \|f_j\|_{r,s_*} \right. \\ & \quad \left. + \sum_i \left(|u|_{D_{m+1};1,s_*} \|f_i\|_{r,s} + \|u\|_{s+1,s} \odot \|f_i\|_{r,s_*} + \|u\|_{r,s_*} \odot \|f_i\|_{s+1,s} \right) \right\}. \end{aligned}$$

The proof of the following lemma is straightforward.

Lemma 3.4. *Let $0 \leq \theta_k^* \leq \frac{1}{(2+k)^2}$. Then*

$$\prod_{k=0}^{\infty} (1 - \theta_k^*) > 1/2.$$

Let $\rho_{k+1} = (1 - \theta_k^)\rho_k$, $\rho_\infty = \lim_{k \rightarrow \infty} \rho_k > 0$. Suppose that $F_k(D_\rho) \subset D_{(1-\theta_k^*)^{-1}\rho}$ for each $\rho \in (0, \rho_k)$. Then $F_i \circ \dots \circ F_0(D_{\rho_\infty/2}) \subset D_{\rho_\infty}$.*

4. FROBENIUS THEOREM WITH PARAMETER AND EXAMPLES

There are several ways to prove the classical Frobenius theorems. In the literature there seems to lack examples showing exact regularity for the various types of Frobenius theorems. In the end of this section, we will provide some examples showing that some regularity results in this paper are almost sharp.

First we reformulate the proof of Frobenius's theorem in F. and R. Nevanlinna [26] for the parametric case. The reader can compare it with the loss of derivatives in Theorem 1.3.

Definition 4.1. Let $[r_1] \geq [r_2] \geq [s]$. The $\mathcal{C}^{r_1, r_2; s}(\overline{U} \times \overline{V}) = \mathcal{C}_{x, y; \lambda}^{r_1, r_2; s}(\overline{U} \times \overline{V})$ is the set of functions $u^\lambda(x, y)$ such that $\partial_x^i \partial_y^j \partial_\lambda^k u^\lambda(x, y)$ are continuous in $\overline{U} \times \overline{V} \times I$ and have bounded $\mathcal{C}^{r_1 - \{r_1\}}(\overline{U})$ norms for $(y, \lambda) \in \overline{V} \times I$, bounded $\mathcal{C}^{r_2 - \{r_2\}}(\overline{V})$ norms for $(x, \lambda) \in \overline{U} \times I$, and bounded $\mathcal{C}^{s - \{s\}}(I)$ norms for $(x, y) \in \overline{U} \times \overline{V}$, for $k \leq s, j + k \leq r_2, i + j + k \leq r_1$.

Proposition 4.2. (Frobenius Theorem with parameter.) *Let r, s satisfy (2.3). Let $S = \{S^\lambda\}$ be a family of (real) Frobenius structures in D of class $\mathcal{C}^{r, s}(\overline{D})$. Assume that S^λ have the constant rank m . Near each point $p \in D$, there exists a family $\{\Phi^\lambda\} \in \mathcal{C}^{r, s}(\overline{U})$ of diffeomorphisms Φ^λ from U onto $B_\delta^M \times B_\epsilon^{N-M}$ such that $\Phi_*^\lambda S^\lambda$ are the span of $\partial_{x_1}, \dots, \partial_{x_M}$. Furthermore, if S_0^λ are tangent to $\mathbf{R}^M \times \{0\}$, then we can take $(\Phi^\lambda)^{-1}: \tilde{x} = x, \tilde{y} = F^\lambda(x, y)$ with $\{F^\lambda\} \in \mathcal{C}^{r+1, r; s}(\overline{B_\delta^M} \times \overline{B_\epsilon^{N-M}})$, where x, y are coordinates of $\mathbf{R}^M, \mathbf{R}^{N-M}$, respectively.*

Proof. We adapt the proof in [26], which does not involve a parameter and is for integers r, s . To recall the proof let us first assume that $S^\lambda = S$ are independent of λ . By a linear change of coordinates, we may assume that $p = 0$ and the tangent space of S_0 is the x subspace. Near the origin, S consists of vector fields annihilated by

$$(4.1) \quad dy_\ell - A_{\ell m} dx_m, \quad 1 \leq \ell \leq N - M.$$

Let d_x be the differential in x variable. We seek integral submanifolds $M: Y = F(x, y)$ so that for a fixed y

$$(4.2) \quad d_x F_\ell(x, y) = A_{\ell m}(x, F(x, y)) dx_m, \quad F(0, y) = y.$$

Then we can verify that $\tilde{x} = x$ and $\tilde{y} = F(x, y)$ transform ∂_{x_m} into $\frac{\partial}{\partial \tilde{x}_m} + A_{\ell m}(\tilde{x}, \tilde{y}) \frac{\partial}{\partial \tilde{y}_\ell}$. It is clear that for the integral manifolds, F must satisfy the equation in the radial integral

$$(4.3) \quad F(x, y) = TF(x, y).$$

Here $(TF)_\ell(x, y) = y_\ell + x_m \int_0^1 A_{\ell m}(tx, F(tx, y)) dt$.

To prove next proposition, let us consider a more general operator

$$(4.4) \quad (TF)_\ell(x, y) := b_\ell(y) + x_m \int_0^1 A_{\ell m}(tx, F(tx, y)) dt$$

with $b_\ell(0) = 0$ and $b_\ell \in \mathcal{C}^r(\overline{B_{\delta_0}^M})$. Let $F_0(x, y) = b(y)$ and $F_{n+1} = TF_n(x, y)$. By the Picard iteration, equations (4.3)-(4.4) admit a \mathcal{C}^1 solution $F = \lim_{n \rightarrow \infty} F_n$ in $K_{\delta, \epsilon} := \overline{B_\delta^M} \times \overline{B_\epsilon^{N-M}}$ for some positive ϵ and δ , provided that on K_{δ_0, ϵ_0}

$$|A(x, y)| < C, \quad |\partial_y A(x, y)| < C.$$

We can verify that F satisfies (4.2). Since (4.4) is a slightly general situation, we provide some details for the reader, using a method in [26, pp. 158-163]. The integrability condition means that

$$d(A_\ell(x, y) dx) = 0 \mod dy_j - A_j(x, y) dx.$$

Substituting $A dx$ for dy in the first identity after the differentiation, we obtain

$$(4.5) \quad \begin{aligned} \partial_{x_i} A_{\ell m}(x, y) - \partial_{x_m} A_{\ell i}(x, y) \\ = \partial_{y_j} A_{\ell i}(x, y) A_{jm}(x, y) - \partial_{y_j} A_{\ell m}(x, y) A_{ji}(x, y). \end{aligned}$$

To show that $d_x F(x, y) = A(x, y) dx$ for $y = F(x, y)$, it suffices to verify that

$$(R_n)_{\ell m}(x, y) = \partial_{x_m} (F_n)_\ell(x, y) - A_{\ell m}(x, F_n(x, y))$$

tends to zero in sup-norm as $n \rightarrow \infty$. Differentiating

$$(F_{n+1})_\ell(x, y) = b_\ell(y) + x_i \int_0^1 A_{\ell i}(tx, F_n(tx, y)) dt,$$

we get

$$\begin{aligned} \partial_{x_m} (F_{n+1})_\ell(x, y) &= \int_0^1 \left\{ A_{\ell m}(tx, F_n(tx, y)) + x_i t (\partial_{x_m} A_{\ell i})(tx, F_n(tx, y)) \right\} dt \\ &\quad + x_i \int_0^1 (\partial_{y_j} A_{\ell i})(tx, F_n(tx, y)) t (\partial_{x_m} (F_n)_j)(tx, y) dt. \end{aligned}$$

Adding $0 = A_{\ell m}(x, F_n(x, y)) - \int_0^1 \partial_t(tA_{\ell m}(tx, F_n(tx, y))) dt$ to the right-hand side, we get

$$\begin{aligned} \partial_{x_m}(F_{n+1})_\ell(x, y) &= A_{\ell m}(x, F_n(x, y)) - x_i \int_0^1 t \partial_{x_i} A_{\ell m}(tx, F_n(tx, y)) dt \\ &\quad - x_i \int_0^1 t \partial_{Y_j} A_{\ell m}(tx, F_n(tx, y)) \partial_{x_i}(F_n)_j(tx, y) dt \\ &\quad + x_i \int_0^1 t (\partial_{x_m} A_{\ell i})(tx, F_n(tx, y)) dt \\ &\quad + x_i \int_0^1 (\partial_{Y_j} A_{\ell i})(tx, F_n(tx, y)) t (\partial_{x_m}(F_n)_j)(tx, y) dt. \end{aligned}$$

We want to express the above in terms of R_n . By (4.5) in which $F_n(x, y)$ substitutes for y , we simplify the right-hand side and obtain

$$\begin{aligned} (R_{n+1})_{\ell m}(x, y) &= x_i \int_0^1 \partial_{Y_j} A_{\ell i}(tx, F_n(tx, y)) (R_n)_{jm}(tx, y) dt - \\ &\quad x_i \int_0^1 \partial_{Y_j} A_{\ell m}(tx, F_n(tx, y)) (R_n)_{ji}(tx, y) dt + A_{\ell m}(x, F_n(x)) - A_{\ell m}(x, F_{n+1}(x, y)). \end{aligned}$$

We have $|\partial_y A(x, y)| < C$ and

$$|A(x, F_{n+1}(x, y)) - A(x, F_n(x, y))| \leq C |F_{n+1}(x, y) - F_n(x, y)|.$$

For some $0 < \theta < 1$, we have $|F_{n+1}(x, y) - F_n(x, y)| \leq C\theta^n$ and

$$|R_{n+1}|_0 \leq C_0 \theta^n + \theta |R_n|_0, \quad n \geq 0.$$

Therefore, R_n tends to 0 in the sup norm as $n \rightarrow \infty$ and $d_x F(x, y) = A(x, F(x, y)) dx$.

We now consider the parametric case. By the initial normalization in Lemma 2.1, we may assume that $p = 0$ and all S_0^λ are tangent to the y -subspace. Thus S^λ are defined by (4.1) in which $A_m = (A_{1m}, \dots, A_{(N-M)m}) = A_m^\lambda$ depend on λ . Then we take $F_0^\lambda(x, y) = b^\lambda(y)$ and $F_{n+1}^\lambda(x, y) = TF_n^\lambda(x, y)$. Thus

$$(4.6) \quad F_{n+1}^\lambda(x, y) = b_\ell^\lambda(y) + x_m \int_0^1 A_{\ell m}^\lambda(\theta x, F_n^\lambda(tx, y)) dt.$$

By the Picard iteration, if $|A|_{K_{\delta_0, \epsilon_0}, 1, 0} < C_0$, we find $F^\lambda = \lim_{n \rightarrow \infty} F_n^\lambda$ with $\{F^\lambda\} \in \mathcal{C}^{1,0}(K_{\delta, \epsilon})$ for some positive δ, ϵ . By a bootstrap argument by shrinking δ, ϵ a few more times and keeping them independent of r, s , one can verify the full regularity. We briefly indicate it. Let $\alpha = \{r\} > 0$ and consider the case $[r] = [s] = 1$ and $\{s\} = \beta \leq \alpha$. For $1 \leq m \leq M$,

$$\begin{aligned} \partial_{x_m} F^\lambda(x, y) &= x_i \int_0^1 t (\partial_{y_\ell} A_i^\lambda)(tx, F^\lambda(tx, y)) \partial_{x_m} F_\ell^\lambda(tx, y) dt \\ &\quad + x_i \int_0^1 t \partial_{x_m} A_i^\lambda(tx, F^\lambda(tx, y)) dt + \int_0^1 A_m^\lambda(tx, F^\lambda(tx, y)) dt. \end{aligned}$$

By another Picard iteration, we can show that $\partial_{x_m} F \in \mathcal{C}^{\alpha,0}$. One can also show that $\{\partial_{y_\ell} F^\lambda\} \in \mathcal{C}^{\alpha,0}$. When $s \geq 1$, we can verify that $\partial_\lambda F^\lambda \in C^0$ and

$$\begin{aligned} \partial_\lambda F^\lambda(x, y) &= \partial_\lambda b^\lambda(y) + x_m \int_0^1 (\partial_{y_\ell} A_m^\lambda)(tx, F^\lambda(tx, y)) \partial_\lambda F_\ell^\lambda(tx, y) dt \\ &\quad + x_m \int_0^1 (\partial_\lambda A_m^\lambda)(tx, F^\lambda(tx, y)) dt. \end{aligned}$$

By the Picard iteration, we can verify that $\{\partial_\lambda F^\lambda\} \in C_*^{\alpha,\beta}$, using $\alpha \geq \beta$.

While taking derivatives one-by-one without further shrinking δ, ϵ , we can verify that $\{F^\lambda\} \in \mathcal{C}^{r,s}$. \square

The above proof for (4.1), (4.3) and (4.4) shows the following.

Proposition 4.3. *Let $\{A^\lambda\} \in \mathcal{C}^{r,s}(\overline{B_{\epsilon_0}^M} \times \overline{B_{\delta_0}^{N-M}})$ with A^λ being $L \times M$ matrices satisfying the integrability condition (4.5). Let $b \in \mathcal{C}^{r,s}(\overline{B_{\epsilon_0}^M})$. Assume that $A^\lambda(0) = 0$ and $b^\lambda(0) = 0$. There exists $\{F^\lambda\} \in \mathcal{C}^{r+1,r;s}(\overline{B_\delta^M} \times \overline{B_\epsilon^{N-M}})$ for some positive δ, ϵ satisfying $F^\lambda(0, y) = b^\lambda(y)$ and*

$$d_x F_\ell^\lambda(x, y) = A_{\ell m}^\lambda(x, F^\lambda(x, y)) dx_m, \quad 1 \leq \ell \leq N - M.$$

It is interesting that the proof by F. and R. Nevanlinna produces a sharp result. Let us demonstrate it by examples. Our first example shows that in general one cannot expect to gain any derivatives in λ for finite smooth case.

Example 4.4. (Complex version.) Let a^λ be a function that is \mathcal{C}^∞ for $\lambda \neq 0$ and is $\mathcal{C}^{1+\alpha}$. Suppose that a^λ is independent of z and $|a^\lambda| < 1$. Let $F^\lambda(z) = z + a^\lambda \bar{z}$. Then $F_*^\lambda \partial_{\bar{z}}$ is

$$Z^\lambda = \partial_{\bar{z}} + \bar{a}^\lambda \partial_z.$$

Thus Z is of class $\mathcal{C}^{1+\alpha}$ in $(z, \lambda) \in \mathbf{C} \times \mathbf{R}$. When regarding $F^\lambda(z)$ as a mapping $\tilde{F}(z, \lambda) = (F^\lambda(z), \lambda)$ and $Z^\lambda(z)$ as a vector field \tilde{Z} in (z, λ) , we still have

$$\tilde{F}_* \partial_{\tilde{z}} = \tilde{Z}.$$

We want to show that there does not exist a diffeomorphism $G: (z, \lambda) \rightarrow G(\lambda, z)$ so that G is defined in a neighborhood of origin in $\mathbf{C} \times \mathbf{R}$ and of class $\mathcal{C}^{1+\beta}$ in (z, λ) for $\beta > \alpha$, while G transform \tilde{Z} into $\partial_{\bar{z}}$.

Suppose that such a G exists. Then $G(F^\lambda(z), \lambda) = H(z, \lambda)$ is holomorphic in z . Write $H(z, \lambda) = (h(z, \lambda), r(z, \lambda))$ and $G(z, \lambda) = (g(z, \lambda), s(z, \lambda))$, where r and s are real valued. Since $r(z, \lambda)$ is holomorphic in z and real-valued, then $r(z, \lambda)$ is independent of z . Thus, $\partial_z h \neq 0$. Set $\tilde{w} = (1 - |a^\lambda|^2)^{-1}(z - a^\lambda \bar{z})$. We have

$$g(z + a^\lambda \bar{z}, \lambda) = h(z, \lambda), \quad g(z, \lambda) = h(\tilde{w}, \lambda).$$

Since $h(z, \lambda)$ is holomorphic in z and is $\mathcal{C}^{1+\alpha}$, Cauchy's formula implies that $\partial_z h(z, \lambda) \in \mathcal{C}^{1+\alpha}$. Then

$$(4.7) \quad \partial_z g(z, \lambda) = (1 - |a^\lambda|^2)^{-1} \partial_{\tilde{w}} h(\tilde{w}, \lambda), \quad \partial_{\bar{z}} g(z, \lambda) = -(1 - |a^\lambda|^2)^{-1} a^\lambda \partial_{\tilde{w}} h(\tilde{w}, \lambda)$$

are in $\mathcal{C}^{1+\alpha}$. Since h is holomorphic in z , then

$$\int_{|z|=\epsilon} z g(z + a^\lambda \bar{z}, \lambda) dz = \int_{|z|=\epsilon} z h(z, \lambda) dz = 0.$$

Let $w = z + a^\lambda \bar{z}$. Taking λ derivative, we get

$$(4.8) \quad A^\lambda \partial_\lambda a^\lambda + B^\lambda \partial_\lambda \bar{a}^\lambda = - \int_{|z|=\epsilon} z \partial_\lambda g(w, \lambda) d\lambda$$

where $A^\lambda = \epsilon^2 \int_{|z|=\epsilon} \partial_w g(w, \lambda) d\lambda$ and $B^\lambda = \int_{|z|=\epsilon} z^2 \partial_{\bar{w}} g(w, \lambda) dz$. The right-hand side and A^λ, B^λ are in \mathcal{C}^β . Since $\partial_w h(0, \lambda) \neq 0$ and $|a^\lambda| < 1$, then (4.7) imply that $|A^\lambda| > |B^\lambda|$, when ϵ is sufficiently small. Solving for $\partial_\lambda a^\lambda, \partial_\lambda \bar{a}^\lambda$ from (4.8), we see that $\partial_\lambda a^\lambda$ is in \mathcal{C}^β , a contradiction.

The next example shows that for Frobenius structures, we cannot expect to gain derivatives. This contrasts with the complex structures for which normalized transformation gain some derivatives.

Example 4.5. (Real version.) For $(x, y, z) \in \mathbf{R}^3$, let $F(x, y, z) = (x, y + a(z)x, z)$. Here a is in $\mathcal{C}^{1+\alpha}$ but not in $\mathcal{C}^{1+\beta}$ for any $\beta > \alpha$. Then $F_* \partial_x$ equals

$$X = \partial_x + a(z) \partial_y.$$

We claim that there is no $G \in \mathcal{C}^{1+\beta}$ with $\beta > \alpha$ transforming X into ∂_x .

Suppose that such a G exists. Then the last two components of $G \circ F(x, y, z) = S(x, y, z)$ are independent of x . Let s, g be last two components of S and G , respectively. We get

$$s(y, z) = g(x, y + a(z)x, z).$$

Then $s(y, z) = g(0, y, z)$ is also in $\mathcal{C}^{1+\beta}$. Since the Jacobian of S does not vanish at the origin, then $\partial_y g(0) \neq (0, 0)$. Without loss of generality, we may assume that $\partial_y g_1(0, 0) \neq 0$. Let $u = u(x, y, z)$ be the $\mathcal{C}^{1+\beta}$ solution to $s_1(x, y) = g_1(x, u, z)$. Then $y + a(z)x = u(x, y, z)$ is in $\mathcal{C}^{1+\beta}$ near $(x, y, z) = 0$, a contradiction.

As mentioned in the introduction, Nirenberg [28] asked if an analytic family of \mathcal{C}^∞ Levi-flat CR vector bundles could be normalized by an analytic family of smooth diffeomorphisms. Nirenberg's question was justified by his theorem on a holomorphic family of complex structures [28]; see Proposition 6.5 below. We now provide a negative answer to the analogous question for the finite smoothness case. This shows a significant difference between the analytic families of general complex Frobenius and complex structures. We recall

$$(4.9) \quad P^\lambda = \partial_x + (|y|^{a+1} + \lambda) \partial_y, \quad (x, y) \in \mathbf{R}^2.$$

Proposition 4.6. *Let $a \in (1, \infty) \setminus \mathbf{N}$. Let P^λ be defined by (4.9). Then $P^\lambda N^\lambda = 0$ does not admit solutions N^λ defined in a neighborhood of the origin in \mathbf{R}^2 that satisfy*

$$(4.10) \quad dN^\lambda(0) \neq 0, \quad \{N^\lambda\} \in \mathcal{C}_*^{b,1}, \quad b > a.$$

In particular, there does not exist a real analytic family of C^b diffeomorphisms that transform P^λ into $\partial_{\hat{x}}$.

Proof. We may assume that $a < b < [a] + 1$. Suppose for the sake of contradiction that there is a family of solutions N^λ satisfying $P^\lambda N^\lambda = 0$ and (4.10). We define $(x, y) = \varphi(\hat{x}, \hat{y})$ as

$$x = \hat{x}, \quad y = \hat{y}(1 - a\hat{x}\hat{y}^a)^{-1/a}.$$

Note that φ preserves the sign of the y component and φ is of class \mathcal{C}^{a+1} . Thus, in what follows, we will restrict y and \hat{y} to be non-negative. Note that $\varphi^{-1}(x, y) = (x, y(1 + axy^a)^{-1/a})$.

Applying the chain rule, we verify that

$$(4.11) \quad \begin{aligned} \varphi_* \partial_{\hat{x}} &= P^0, \quad (\varphi)_*^{-1} \partial_y = Q, \\ Q &:= (1 - a\hat{x}\hat{y}^a)^{(a+1)/a} \partial_{\hat{y}}. \end{aligned}$$

Define $N_j = (\partial_\lambda)^j|_{\lambda=0} N^\lambda$. Using $P^\lambda N^\lambda = 0$, the constant and linear terms in λ on the right side yield

$$P^0 N_0 = 0, \quad P^0 N_1 = -\partial_y N_0.$$

In the (\hat{x}, \hat{y}) -coordinates, the above equations become

$$\partial_{\hat{x}} \hat{N}_0 = 0, \quad \partial_{\hat{x}} \hat{N}_1 = -Q \hat{N}_0$$

with $\hat{N}_i = N_i \circ \varphi \in C^b$. Let us drop hats in \hat{N}_i and (\hat{x}, \hat{y}) . We first note that $N_0(x, y)$ is independent of x , which is now denoted by $B(y)$. We have $B \in C^b$. By the fundamental theorem of calculus, we obtain

$$(4.12) \quad N_1(x, y) - N_1(0, y) = -B'(y)u(x, y),$$

$$(4.13) \quad u(x, y) := \int_0^x (1 - axy^a)^{(a+1)/a} dx.$$

We have $u \in C^a$. Fix a small $x_0 \neq 0$ so that $u(x_0, y) \neq 0$. Since $b > a$, by $B'(y) = (N_1(x_0, y) - N_1(0, y))u^{-1}(x_0, y)$ we obtain $B' \in C^a$.

Next, we show that $yB'(y)$ has a better regularity. Differentiating (4.12), we obtain

$$B^{([a]+1)}(y)u(x, y) = -B'(y)\partial_y^{[a]}u(x, y) + \tilde{B}(x_0, y),$$

with $\tilde{B} \in C^{b-[a]}$. It is straightforward that $y\partial_y^{[a]}u(x, y)$ is C^1 in y . Therefore, $yB^{([a]+1)}(y)$ is in $C^{b-[a]}$. Now, the function $yB'(y)$ is in C^b because $(yB'(y))^{[a]} = yB^{[a]+1} + [a]B^{[a]}(y)$.

Using the Taylor series of the power function, we express (4.13) as

$$u(x, y) = x - \frac{a+1}{2}x^2y^a + yE(x, y),$$

with $E \in C^{2a-1}$. In (4.12), we substitute u by the above expansion and obtain

$$xB'(y) - \frac{a+1}{2}x^2y^aB'(y) = N_1(0, y) - N(x, y) - (yB'(y))E(x, y).$$

The right-hand side is of class $C^{b'}$ for $b' = \min(2a-1, b)$. Plugging in $x = \epsilon, 2\epsilon$ in the above identity, we get two equations for $B'(y)$ and $y^aB'(y)$. Solving them, we conclude that $B'(y), y^aB'(y)$ are in $C^{b'}$. Since $dN^0(x, y) \neq 0$ and $B(y) = N_0(x, y)$, then $B'(0) \neq 0$. Therefore, y^a is in $C^{b'}$, a contradiction. \square

We can also prove the following result for non-homogeneous equations.

Proposition 4.7. *Let P^λ be defined by (4.9). Let $a' \in (1, \infty) \setminus \mathbf{N}$ and $a' < a$. Then*

$$P^\lambda v^\lambda = |y|^{a'+1}$$

does not admit a solution $\{v^\lambda\} \in \mathcal{C}_^{b,1}(\overline{D})$ for $b > a'$ and any neighborhood D of 0.*

Proof. Assume for the contrary that such a solution $\{v^\lambda\}$ exists. Let $(x, y) = \varphi(\hat{x}, \hat{y})$ be as in the above proof, and let $v_i = \partial_{\hat{\lambda}}^i|_{\hat{\lambda}=0} v^\lambda$ and $\hat{v}_i(\hat{x}, \hat{y}) = v_i \circ \varphi(\hat{x}, \hat{y})$. Then $P^\lambda v^\lambda = |y|^{a'+1}$ implies that

$$P^0 v_0 = |y|^{a'+1}, \quad P^0 v_1 = -\partial_y v_0.$$

Let us drop hats in $\hat{v}_i, \hat{x}, \hat{y}, \hat{v}_i$, and restrict $y \geq 0$. By (4.11), we get

$$\partial_x v_0(x, y) = y^{a'+1}(1 - axy^a)^{-(a'+1)/a}, \quad \partial_x v_1(x, y) = -(1 - axy^a)^{(a+1)/a} \partial_y v_0(x, y).$$

Using Taylor series of the power function in both identities, we get

$$\begin{aligned} v_0(x, y) &= A(y) + xy^{a'+1} + E_{2a'+1}(x, y), \\ v_1(x, y) &= B(y) - C(x, y)A'(y) - \frac{a'+1}{2}x^2y^{a'} + E_{2a'}(x, y), \end{aligned}$$

where $C(x, y) = x + x^2E_a(x, y)$, and $E_r \in \mathcal{C}^r$. We may assume that $a' < b < [a'] + 1$ and $b < a$. Since $\{v^\lambda\} \in \mathcal{C}_*^{b,1}$ and $\varphi \in \mathcal{C}^{a+1}$, then v_i, A, B are in \mathcal{C}^b . Plugging in $x = \epsilon, 2\epsilon$, we can express $A'(y), y^{a'}$ as linear combinations of \mathcal{C}^b functions, of which the coefficients are also \mathcal{C}^b functions. Here we have used

$$\begin{vmatrix} (2\epsilon)^2 & \epsilon^2 \\ C(2\epsilon, y) & C(\epsilon, y) \end{vmatrix} = 2\epsilon^3 + 4\epsilon^4 E_a(\epsilon, y) - 4\epsilon^4 E_a(4\epsilon, y) \neq 0$$

when $\epsilon > 0$ is sufficiently small. This shows that $y^{a'}$ is of class \mathcal{C}^b with $b > a'$, which is a contradiction. \square

By Proposition 4.2, the equation in Proposition 4.7 has a solution $\{v^\lambda\} \in \mathcal{C}^{r,r}(D)$ for some neighborhood D of the origin and $r = a' + 1$, when $a \geq a'$ and $a' \in (0, \infty] \setminus \mathbf{N}$.

5. A HOMOTOPY FORMULA FOR MIXED REAL AND COMPLEX DIFFERENTIALS

In this section, we will adapt estimates for the Koppelman-homotopy formula, which are due to Webster [31]. We will also recall a homotopy formula for the $d^0 + \bar{\partial}$ complex, which is defined in this section. The homotopy formula for $d^0 + \bar{\partial}$ is obtained by combining the Bochner-Martinelli and Koppelman-Leray formulae for all degrees and Poincaré's homotopy formula, which is due to Treves [30].

Let us first recall the Bochner-Martinelli and Koppelman-Leray formulae for the ball $B_\rho^{2n} \subset \mathbf{C}^n$. The reader is refer to Webster [31] and Chen-Shaw [4] for details. For $(\zeta, z) \in \mathbf{C}^n \times \mathbf{C}^n$ with $\zeta \neq z$, let

$$\begin{aligned} \omega^0 &= \frac{1}{2\pi i} \frac{(\bar{\zeta} - \bar{z}) \cdot d\zeta}{|\zeta - z|^2}, \quad \omega^1 = \frac{1}{2\pi i} \frac{\bar{\zeta} \cdot d\zeta}{\bar{\zeta} \cdot (\zeta - z)}, \\ \Omega^\ell &= \omega^i \wedge (\bar{\partial}\omega^i)^{n-1}, \quad i = 0, 1; \\ \Omega^{01} &= \omega^0 \wedge \omega^1 \wedge \sum_{\alpha+\beta=n-2} (\bar{\partial}\omega^0)^\alpha \wedge (\bar{\partial}\omega^1)^\beta. \end{aligned}$$

Let $\Omega_q^0, \Omega_q^{01}$ be the components of Ω^0 and Ω^{01} of type $(0, q)$ in z , respectively:

$$\Omega^i = \sum_{q=0}^{n-1} \Omega_q^i, \quad i = 0, 1; \quad \Omega^{01} = \sum_{q=0}^{n-2} \Omega_q^{01}.$$

The Koppelman lemma says that $\bar{\partial}_{\zeta,z}\Omega^0 = 0$ and $\bar{\partial}_{\zeta,z}\Omega^{01} = \Omega^0 - \Omega^1$. Thus

$$\bar{\partial}_\zeta \Omega_q^0 = -\bar{\partial}_z \Omega_{q-1}^0, \quad \Omega_q^0 = \Omega_q^1 - \bar{\partial}_z \Omega_{q-1}^0 - \bar{\partial}_\zeta \Omega_q^{01}.$$

Let φ be a $(0, q)$ form in $\mathcal{C}^1(\overline{B_\rho^{2n}})$, and let f be a \mathcal{C}^1 function. We have

$$\begin{aligned}\varphi &= \bar{\partial}P_q\varphi + P_{q+1}\bar{\partial}\varphi, \quad q > 0; \quad \varphi = B_0^1\varphi + P_1\bar{\partial}\varphi, \quad q = 0; \\ P_q\varphi &= P_q^0\varphi - P_q^{01}\varphi; \\ P_q^0\varphi(z) &:= \int_{B_\rho^{2n}} \Omega_{q-1}^0(\zeta, z) \wedge \varphi(\zeta), \quad P_q^{01}\varphi(z) := \int_{\partial B_\rho^{2n}} \Omega_{q-1}^{01}(\zeta, z) \wedge \varphi(\zeta), \quad q > 0; \\ B_0^1\varphi(z) &:= \int_{\partial B_\rho^{2n}} \Omega_0^1(\zeta, z) \varphi(\zeta), \quad q = 0.\end{aligned}$$

Note that $\bar{\partial}_z\Omega_0^1(\zeta, z) = 0$. When $n = 1$, $P_1 = P_1^0$ is the Cauchy-Green operator.

The Poincaré lemma for a q -form ϕ on the ball $\overline{B_\rho^M}$ has the form

$$(5.1) \quad \begin{aligned}\phi &= d^0R_q\phi + R_{q+1}d^0\phi, \quad q > 0; \quad \phi = R_1d^0\phi + \phi(0), \quad q = 0; \\ R_q\phi(t) &:= \int_{\theta \in [0,1]} H^*\phi(t, \theta).\end{aligned}$$

Here $H(t, \theta) = \theta t$ for $(t, \theta) \in B_\rho^M \times [0, 1]$. Note that

$$d(\theta t_1) \wedge \cdots \wedge d(\theta t_q) = \sum (-1)^{j-1} t_j \theta^{q-1} d\theta \wedge dt_1 \wedge \cdots \wedge \widehat{dt_j} \cdots \wedge dt_q.$$

It is immediate that for $q > 0$

$$(5.2) \quad |R_q\phi|_{\rho;a} \leq C_a \rho |\phi|_{\rho;a}, \quad \phi \in \mathcal{C}^a(\overline{B_\rho^M}), \quad a \in [0, \infty).$$

Note that there is no gain in derivatives in estimates of R_q . Set $R_j = 0$ for $j > M$.

Next, we consider the complex for the exterior differential

$$\mathcal{D} := d_t + \bar{\partial}_z$$

for $(z, t) \in \mathbf{C}^n \times \mathbf{R}^M$. We will also write the above as $\mathcal{D} = d^0 + \bar{\partial}$. A differential form φ is called of *mixed type* $(0, q)$ if

$$\varphi = \sum_{i=0}^q [\varphi]_i,$$

where $[\varphi]_i = \sum_{|I|=i, |I|+|J|=q} a_{IJ} d\bar{z}^I \wedge dt^J$. Thus

$$[\varphi]_i = 0, \quad \text{for } i > n.$$

The \mathcal{D} acts on a function f and a $(0, q)$ form as follows

$$\mathcal{D}f = \sum \frac{\partial f}{\partial t_m} dt_m + \sum \frac{\partial f}{\partial \bar{z}_\alpha} d\bar{z}_\alpha, \quad \mathcal{D} \sum a_{IJ} d\bar{z}^I \wedge dt^J = \sum \mathcal{D}a_{IJ} \wedge d\bar{z}^I \wedge dt^J.$$

Let $\mathcal{C}_{0,q}^a$ be space of $(0, q)$ -forms of which the coefficients are of class \mathcal{C}^a . Then

$$\mathcal{D}: \mathcal{C}_{(0,q)}^a \rightarrow \mathcal{C}_{(0,q+1)}^{a-1}.$$

From $(d^0 + \bar{\partial} + \partial)^2 = 0$, we get

$$\mathcal{D}^2 = 0, \quad d^0\bar{\partial} + \bar{\partial}d^0 = 0.$$

We also have

$$[\mathcal{D}\varphi]_0 = d^0[\varphi]_0, \quad [\mathcal{D}\varphi]_i = d^0[\varphi]_{i+1} + \bar{\partial}[\varphi]_i, \quad 0 < i \leq n.$$

For $\varphi = \sum \varphi_{IJ} d\bar{z}^I \wedge dt^J = \sum \tilde{\varphi}_{IJ} dt^J \wedge d\bar{z}^I$ on $\overline{B_\rho^{2n}} \times \overline{B_\rho^M}$, define

$$\begin{aligned} P_i \varphi &= \sum_{|I|=i} P_i(\varphi_{IJ} d\bar{z}^I) \wedge dt^J, \\ R_{q-i} \varphi &= \sum_{|J|=q-i} R_{q-i}(\tilde{\varphi}_{IJ} dt^J) \wedge d\bar{z}^I. \end{aligned}$$

Thus $P_i \varphi = P_i[\varphi]_i$, while $R_{q-i} \varphi = R_{q-i}[\varphi]_i$ if φ has the (mixed) type $(0, q)$. We have

$$\begin{aligned} d^0 P_i \varphi &= dt_m \wedge \partial_{t_m} P_i(\varphi_{IJ} \wedge d\bar{z}^I) \wedge dt^J \\ &= (-1)^{i-1} \partial_{t_m} P_i(\varphi_{IJ} \wedge d\bar{z}^I) \wedge dt_m \wedge dt^J. \end{aligned}$$

Also $\partial_{t_m} P_i(\varphi_{IJ} d\bar{z}^I) \wedge dt_m = P_i(\partial_{t_m} \varphi_{IJ} d\bar{z}^I \wedge dt_m)$. Thus

$$(5.3) \quad d^0 P_i \varphi = -P_i d^0 \varphi \quad i > 0; \quad \bar{\partial} R_{q-i} \varphi = -R_{q+1-i} \bar{\partial} \varphi, \quad i < q.$$

We now derive the following homotopy formulae for \mathcal{D} .

Lemma 5.1. *Let $1 \leq q \leq N$. Let $D_\rho = B_\rho^{2n} \times B_\rho^M$. Let φ be a mixed $(0, q)$ form in \overline{D}_ρ of class $\mathcal{C}^{1,0}(\overline{D}_\rho)$. Then we have two homotopy formulae*

$$(5.4) \quad \varphi = \mathcal{D}T_q \varphi + T_{q+1} \mathcal{D} \varphi,$$

$$(5.5) \quad \varphi = \mathcal{D}\tilde{T}_q \varphi + \tilde{T}_{q+1} \mathcal{D} \varphi,$$

$$(5.6) \quad T_q \varphi(z, t) = P_q[\varphi]_q(\cdot, 0)(z) + \sum_{i < q} R_{q-i}[\varphi]_i(z, \cdot)(t),$$

$$(5.7) \quad \tilde{T} \varphi = R_q B_0^1[\varphi]_0 + \sum_{i > 0} P_i[\varphi]_i.$$

Proof. The formula (5.7) is derived in Treves [30, VI.7.12, p. 294] for $q \leq n$, while (5.6) is stated in [30, VI.7.13, p. 294] for $q \leq m$. For the convenience of the reader, we derive them for all q .

Let us start with the integral representation of $[\varphi]_0$ in $|z| < r$. Since φ has total degree q , then $\deg_x[\varphi]_0 = q > 0$. We apply the Bochner-Martinelli formula for functions and the Poincare formula for d^0 to obtain

$$(5.8) \quad [\varphi]_0 = B_0^1[\varphi]_0 + P_1 \bar{\partial}[\varphi]_0 = (d^0 R_q B_0^1[\varphi]_0 + R_{q+1} d^0 B_0^1[\varphi]_0) + P_1 \bar{\partial}[\varphi]_0.$$

Since $\bar{\partial}_z B_0^1(\zeta, z) = 0$, we have $d^0 R_q B_0^1[\varphi]_0 = \mathcal{D} R_q B_0^1[\varphi]_0$. Combining with $d^0[\varphi]_0 = [\mathcal{D}\varphi]_0$, we express (5.8) as

$$(5.9) \quad [\varphi]_0 = \mathcal{D} R_q B_0^1[\varphi]_0 + R_{q+1} B_0^1[\mathcal{D}\varphi]_0 + P_1 \bar{\partial}[\varphi]_0.$$

Assume that $j > 0$. By the Koppelman-Leray formula, we obtain

$$(5.10) \quad [\varphi]_j = \bar{\partial} P_j[\varphi]_j + P_{j+1} \bar{\partial}[\varphi]_j = \mathcal{D} P_j[\varphi]_j - d^0 P_j[\varphi]_j + P_{j+1} \bar{\partial}[\varphi]_j.$$

By (5.3) and $d^0[\varphi]_j = [\mathcal{D}\varphi]_j - \bar{\partial}[\varphi]_{j-1}$, we obtain

$$\begin{aligned} \sum_{j > 0} (-d^0 P_j[\varphi]_j + P_{j+1} \bar{\partial}[\varphi]_j) &= \sum_{j > 0} (P_j[\mathcal{D}\varphi]_j - P_j \bar{\partial}[\varphi]_{j-1} + P_{j+1} \bar{\partial}[\varphi]_j) \\ &= -P_1 \bar{\partial}[\varphi]_0 + \sum_{j > 0} P_j[\mathcal{D}\varphi]_j. \end{aligned}$$

Here we have used $P_{n+1} = 0$. Using (5.9) and (5.10), we obtain

$$\varphi = \mathcal{D}R_q B_0^1[\varphi]_0 + R_{q+1} B_0^1[\mathcal{D}\varphi]_0 + \sum_{j>0} \mathcal{D}P_j[\varphi]_j + \sum_{j>0} P_j[\mathcal{D}\varphi]_j,$$

which gives us (5.5) and (5.7).

Suppose that φ has the mixed type $(0, q)$. By the Poincaré lemma we obtain

$$\varphi = [\varphi]_q + \sum_{i<q} (d^0 R_{q-i}[\varphi]_i + R_{q+1-i} d^0[\varphi]_i) = [\varphi]_q + \sum_{i<q} \mathcal{D}R_{q-i}[\varphi]_i + R_{q+1-i} \mathcal{D}[\varphi]_i.$$

Here we have used $\bar{\partial}R_{q-i}[\varphi]_i = -R_{q+1-i}\bar{\partial}[\varphi]_i$ for $i < q$ by (5.3). We express

$$\sum_{i<q} R_{q+1-i} \mathcal{D}[\varphi]_i = \sum_{i<q} R_{q+1-i} ([d^0\varphi]_i + [\bar{\partial}\varphi]_{i+1}) = -R_1[d^0\varphi]_q + \sum_{i<q} R_{q+1-i}[\mathcal{D}\varphi]_{i+1},$$

because $[\bar{\partial}\varphi]_0 = 0$. We have

$$[\varphi]_q(z, t) - R_1 d^0[\varphi]_q(z, \cdot)(t) = [\varphi]_q(z, 0).$$

We now apply the Koppelman-Leray formula to express

$$[\varphi]_q(\cdot, 0) = \bar{\partial}P_q[\varphi]_q(\cdot, 0) + P_{q+1}\bar{\partial}[\varphi]_q(\cdot, 0) = \mathcal{D}P_q[\varphi]_q(\cdot, 0) + P_{q+1}([\mathcal{D}\varphi]_{q+1}(\cdot, 0)).$$

Combining the identities, we get (5.4) and (5.6). \square

We first consider the case that $N = 2n$. We recall some estimates for the homotopy formulae in Webster [31], adapting them for the parametric version. Note that $P_q^0\varphi$ are integrals of the potential-theoretic type

$$L_0 f(x) = \int_{B_\rho^{2n}} \partial_{\xi_i} p(\xi, x) f(\xi) dV(\xi),$$

where $\xi = (\operatorname{Re} \zeta, \operatorname{Im} \zeta)$ and $x = (\operatorname{Re} z, \operatorname{Im} z)$, $p(\xi, x) = |\xi - x|^{2-2n}$ and f are real or imaginary part of the coefficients of φ .

We first consider the case $\rho = 1$. The general case will be achieved by a dilation. It is a classical result that

$$|L_0 f|_{1;\alpha} \leq C_\alpha |f|_{1;0}, \quad 0 < \alpha < 1.$$

Thus, we have

$$(5.11) \quad |\{L_0 f^\lambda\}|_{1;\alpha,s} \leq C_\alpha |\{f^\lambda\}|_{1;0,\alpha}, \quad 0 < \alpha < 1.$$

Using a smooth cut-off function χ which equals 1 in a domain $B_{1-\theta/2}^{2n}$. We also assume that χ has compact support in $B_{1-\theta/4}^{2n}$ and $|\chi|_r \leq C_r \theta^{-r}$. Decompose $f = f_0 + f_1$ for $f_0 = \chi f$. Then

$$\|f_0\|_{1;r,s} \leq \frac{C_r}{\theta^r} \|f\|_{1;r,s}.$$

We can also write

$$L_0 f_0(x) = \int_{B_2^{2n}} \partial_{\xi_i} p(\xi, x) f_0(\xi) dV(\xi).$$

By a classical estimate for the Hölder norm (see Gilbarg-Trudinger [7, Lemmas 4.1 and 4.4]), we obtain

$$\begin{aligned} |L_0 f_0|_{1;r+1} &\leq |L_0 f_0|_{2;r+1} \leq C_r |f_0|_{1;r}, \quad r \in (0, \infty) \setminus \mathbf{N}, \\ |L_0 f_0|_{1;r+1} &\leq |L_0 f_0|_{2;r} \leq C_r |f_0|_{1;r}, \quad r \in (0, \infty) \setminus \mathbf{N}. \end{aligned}$$

Here C_r depends on $1/\{r\}$ also. When f^λ is a family of functions on D , we decompose $f^\lambda = f_0^\lambda + f_1^\lambda$ with $f_0^\lambda = \chi f^\lambda$. By the linearity of L_0 , we can also estimate derivatives in λ easily. We have

$$|\{L_0 f_0^\lambda\}|_{1;r+1,s} \leq \frac{C_r}{\theta^r} |\{f^\lambda\}|_{1;r,s}, \quad s \in \mathbf{N}, \quad r \in \mathbf{R} \setminus \mathbf{N}, \quad M = 0.$$

In particular, combining with (5.11), we can obtain

$$|\{L_0 f_0^\lambda\}|_{1;r,s} \leq \frac{C_r}{\theta^{[r]}} |\{f^\lambda\}|_{1;[r],s}, \quad s \in \mathbf{N}, \quad r \in \mathbf{R}.$$

Without gaining derivatives, it is straightforward that

$$|\{L_0 f_0^\lambda\}|_{1;r,s} \leq \frac{C_r}{\theta^{[r]}} |\{f^\lambda\}|_{1;r,s}, \quad r, s \in \mathbf{R}.$$

For $L_0 f_1^\lambda$, we differentiate the integrand directly and obtain

$$|\{L_0 f_1^\lambda\}|_{1-\theta;r+1,s} \leq \frac{C_r}{\theta^r} |\{f_1^\lambda\}|_{1;0,s} \leq \frac{C_r}{\theta^r} |\{f^\lambda\}|_{1;0,s}, \quad s \in \mathbf{N}, \quad r \in (0, \infty) \setminus \mathbf{N}, \quad M = 0.$$

Here we have used, for $|z| \leq 1 - \theta$ and $r \notin \mathbf{N}$,

$$\int_{1-\theta/2 < |\zeta| < 1} \frac{1}{|\zeta - z|^{2n+r}} dV(\zeta) \leq \frac{C_r}{\theta^r}, \quad r \in (0, \infty) \setminus \mathbf{N}.$$

The boundary term $P_{q-1}^{01} \varphi$ in the homotopy formula is a sum of

$$L_1 f^\lambda(z) = \int_{\partial B_1^{2n}} \frac{(\zeta_i - z_i) \bar{\zeta}_j f^\lambda(\zeta)}{|\zeta - z|^{2(n-1-k)} (\bar{\zeta} \cdot (\zeta - z))^{k+1}} dS(\zeta), \quad 0 \leq k \leq n-2,$$

where f^λ is a coefficient of φ^λ , and dS is a $(n, n-1)$ -form in ζ with constant coefficients. When $z \in B_1^{2n}$ and $\zeta \in \partial B_1^{2n}$ are close to a boundary point in ∂B_1^{2n} , we may choose local coordinates $s = (s_1, s') \in \mathbf{R}^{2n-1}$ so that $|\bar{\zeta} \cdot (\zeta - z)| \geq (\delta + |s_1| + |s|^2)/C$ and $|\zeta - z| \geq (|s| + \delta)/C$ for $\delta = 1 - |z|$. Then an $[r]$ -th order derivatives of $L_1 f^\lambda$ is a linear combination of

$$L_1^K(z) = \int_{\partial B_1^{2n}} \frac{(\zeta - z, \bar{\zeta} - \bar{z})^K p_K(\zeta) f^\lambda(\zeta)}{|\zeta - z|^{2(n-1-k+m_1)} (r_\zeta \cdot (\zeta - z))^{k+1+m_2}} dS(\zeta)$$

where $|K| = m_1 + 1$ and $[r] = m_1 + m_2$. Let $g(\zeta, z)$ be the integrand in L_1^K . Let $\zeta \in \partial B_1^{2n}$. For $|z| < 1 - \theta$, we have

$$\begin{aligned} |g(\zeta, z)| &\leq \frac{C|f|_{1;0,0}}{|\zeta - z|^{2(n-k)-3+m_1} |r_\zeta \cdot (\zeta - z)|^{k+1+m_2}} \\ &\leq \frac{C\theta^{-[r]}|f|_{1;0,0}}{|\zeta - z|^{2(n-k)-3} |r_\zeta \cdot (\zeta - z)|^{k+1}} \\ &\leq \frac{C'\theta^{-[r]}|f|_{1;0,0}}{|s|^{2n-3}(|s_1| + |s'|^2)}. \end{aligned}$$

Also $\int_0^1 \int_0^1 \frac{r^{2n-3}}{(s_1+r)^{2n-3}(s_1+r^2)} ds_1 dr < \infty$. This shows that $|L_1|_{1-\theta;[r],s} \leq C\theta^{-[r]}|f|_{1;0,s}$. For the Hölder ratio in z variables, let $\alpha = \{r\} > 0$ and $z_1, z_0 \in B_{1-\theta}^{2n}$. If $|z_1 - z_0| > \theta$, we get $|L_1^K(z_1) - L_1^K(z_0)| \leq (|L_1^K(z_1)| + |L_1^K(z_0)|)|z_1 - z_0|^\alpha / \theta^\alpha \leq C|z_1 - z_0|^{\alpha\theta^{-[r]-\alpha}}|f|_{1;0,0}$. Assume

that $|z_1 - z_0| \leq \theta$, and let $z(\lambda) = (1 - \lambda)z_0 + \lambda z_1$. We still have $1 - |z(\lambda)| \geq \theta$. As we just proved, the gradient $\nabla g(\zeta, z)$ in the z -variables satisfies

$$|\nabla_z g(\zeta, z(\lambda))| \leq C' \theta^{-[r]-1} |f|_{1;0,0} |s|^{-2n+3} (|s_1| + |s'|^2)^{-1}.$$

We have $|z_2 - z_1| \leq |z_1 - z_1|^\alpha \theta^{1-\alpha}$. Then by the gradient estimate and integration, we obtain

$$|P_q^{01} \varphi|_{1-\theta; r+1, s} \leq C_r \theta^{-r} |\varphi|_{1;0,s}, \quad s \in \mathbf{N}.$$

Without gaining derivatives, we can also verify

$$|P_q^{01} \varphi|_{1-\theta; r, s} \leq C_r \theta^{-[r]} |\varphi|_{1;0,s}, \quad r, s \in [0, \infty).$$

We now consider the estimates on B_ρ^{2n} with $\rho < C$. Let $z = \tau(\tilde{z}) = \rho \tilde{z}$. Then $\tau_\rho: B_1^{2n} \rightarrow B_\rho^{2n}$. For a $(0, q)$ form φ_q , we have

$$|\tau_\rho^* \varphi_q|_{\rho'; r, s} \leq C_r \rho^q |\varphi_q|_{\rho'; r, s}, \quad |(\tau_\rho^{-1})^* \tilde{\varphi}_q|_{\rho'; r, s} \leq C_r \rho^{-q-r} |\tilde{\varphi}_q|_{\rho'; r, s}, \quad \rho < C.$$

Note that $\tau_\rho \times \tau_\rho$ preserves Ω_q^0 and Ω_q^{01} . Now we get

$$P_{B_\rho^{2n}} = (\tau_\rho^{-1})^* P_{B_1^{2n}} \tau_\rho^*.$$

Then we have

$$\begin{aligned} |P_{B_\rho^{2n}} \varphi_q|_{1-\theta; r+1, s} &\leq C_r \rho^{-q+1-r-1} |P_{B_1^{2n}} \tau_\rho^* \varphi_q|_{1-\theta; r+1, s} \leq C_r \rho^{-r} \theta^{-r} |\varphi_q|_{\rho; r, s}, \\ |P_{B_\rho^{2n}} \varphi_q|_{1-\theta; r, s} &\leq C_r \rho^{-q+1-r} |P_{B_1^{2n}} \tau_\rho^* \varphi_q|_{1-\theta; r, s} \leq C_r \rho^{1-r} \theta^{-[r]} |\varphi_q|_{\rho; [r], s}. \end{aligned}$$

In summary we have obtain, for $0 < \rho < C$,

$$(5.12) \quad |P_q \varphi|_{(1-\theta)\rho; r+1, s} \leq C_r (\rho \theta)^{-r} |\varphi|_{\rho; r, s}, \quad r \notin \mathbf{N}, \quad s \in \mathbf{N}.$$

The above estimate gains one derivative in z -variables. Without the gain, we obtain for $0 < \rho < 2$

$$(5.13) \quad |P_q \varphi|_{(1-\theta)\rho; r, s} \leq C_r \rho^{1-r} \theta^{-[r]} |\varphi|_{\rho; [r], s}, \quad r, s \in [0, \infty).$$

Note that the decomposition $\varphi = \sum [\varphi]_j$ does not commute with \mathcal{D} . Obviously, the decomposition respects the estimates; namely

$$|\varphi|_{\rho; r, s} = \max_j |[\varphi]_j|_{\rho; r, s}.$$

Therefore, we also have

$$|T\varphi|_{\rho; r, s} \leq \sum_j |P_j [\varphi]_j|_{\rho; r, s} + |R_q B_0^1 [\varphi]_0|_{\rho; r, s}.$$

Combining two estimates (5.13) and (5.2), we have the following.

Proposition 5.2. *Let $D_\rho = B_\rho^{2n} \times B_\rho^M \subset \mathbf{C}^n \times \mathbf{R}^M$. Let φ be a differential form on \overline{D}_ρ of mixed type $(0, q)$ with $1 \leq q \leq n + m$. Let $\varphi \in \mathcal{C}_*^{r, s}(\overline{D}_\rho)$. We have*

$$(5.14) \quad \begin{aligned} |T_q \varphi|_{D_{(1-\theta)\rho}; r+1, s} &\leq C_r (\rho \theta)^{-r} |\varphi|_{D_\rho; r, s}, \quad s \in \mathbf{N}, \quad r \in (s, \infty) \setminus \mathbf{N}, \quad M = 0; \\ |T_q \varphi|_{D_{(1-\theta)\rho}; r, s} &\leq C_r \rho^{1-r} \theta^{-[r]} |\varphi|_{D_\rho; [r], s}, \quad r, s \in [0, \infty). \end{aligned}$$

Assume further that $\mathcal{D}\varphi = 0$. Then $\mathcal{D}T_q \varphi = \varphi$.

By estimating in the $C_*^{r, s}$ norm on $B_\rho^{2n} \times B_\rho^M$, we have taken the advantage that the chain rule is not used in the construction of the homotopy formula.

6. THE PROOF FOR A FAMILY OF COMPLEX STRUCTURES

In this section, we will present our first rapid iteration proof for the special case. The arguments do not involve the Nash-Moser smoothing methods, due to the gain of a full derivative in the estimates for the Koppelman-Leray homotopy formula. However, introducing parameter requires an additional argument to obtain rapid iteration in higher order derivatives via the rapid convergence in low order derivatives. This include the \mathcal{C}^∞ case. We will use the rapid methods from Gong-Webster [9, 11].

We will apply rapid iteration methods several times to prove our results. Therefore, it will be convenient to proof a general statement for the rapid iteration.

Definition 6.1. Following Webster [33] we say that the sequence L_j grows (at most) *linearly* if

$$(6.1) \quad 0 \leq L_j \leq e^{P(j)}, \quad j \geq 0$$

for some polynomial P of non-negative coefficients. We say that e_j *converges rapidly* (to zero), if

$$(6.2) \quad 0 \leq e_j \leq \hat{e}_0 b^{\kappa^j - 1}, \quad \kappa > 1, \quad 0 < b < 1, \quad j \geq 0.$$

A precise notion is that the L_j grows at most *exponentially* and the e_j converges to 0 *double exponentially*. Fix a positive constant a . Obviously, if L_j grows linearly, so does L_j^a . And e_j^a converges rapidly with e_j .

Lemma 6.2. Let L_j, P be as in (6.1), and let e_j, κ, b be as in (6.2).

- (i) Let Q be a real polynomial in $t \in \mathbf{R}$. There exists $\hat{e}'_0 > 0$, which depends only on κ, b, Q , so that if $\hat{e}_0 \leq \hat{e}'_0$ additionally, then

$$e_j \leq e^{-Q(j)}, \quad j \geq 0.$$

- (ii) Suppose that M_j is a sequence of non-negative numbers satisfying

$$M_{j+1} \leq L_j M_j, \quad j \geq 0.$$

Then $M_{j+1} \leq M_0 e^{P_1(j)}$ for $P_1(j) = P(0) + \dots + P(j)$ when $j \geq 0$, where P is in (6.1).

Set $P(-1) := 0$. In particular, M_j grows linearly.

- (iii) Suppose that a_j is a sequence of non-negative numbers such that

$$0 \leq a_{j+1} \leq L_j e_j a_j.$$

Let $b_1 > b^{1/(\kappa-1)}$ for b in (6.2). There is a constant C , depending only on \hat{e}_0, b, b_1 , and P , such that

$$a_j \leq C a_0 b_1^{\kappa^j}.$$

In particular a_j converges rapidly.

Proof. (i). Obviously, there exists j_0 so that $b^{\kappa^j - 1} < e^{-Q(j)}$ for $j > j_0$. Choose $0 < \hat{e}'_0 < 1$ sufficiently small so that $\hat{e}'_0 b^{\kappa^j - 1} \leq e^{-Q(j)}$ for $0 \leq j \leq j_0$. We get (i). (ii) is immediate.

- (iii) We have

$$a_{j+1} \leq a_0 \hat{e}'_0 \prod_{i=0}^{j-1} (b^{\kappa^i - 1} e^{P(i)}) \leq a_0 \hat{e}'_0 e^{P_1(j-1)} b^{\frac{\kappa^j - 1}{\kappa - 1} - j}.$$

Since $b_1 > b^{1/(\kappa-1)}$, we can find j_0 so that for $j > j_0$,

$$\hat{e}_0^j e^{P_1(j)} b^{\frac{\kappa^j-1}{\kappa-1}-j} < b_1^{\kappa^j}.$$

Take $C > 1$ so that the above left side is less than $Cb_1^{\kappa^j}$ for $0 \leq j \leq j_0$. \square

The following is one of main ingredients in the KAM rapid iteration procedures.

Proposition 6.3. *Let K_j be two sequences of numbers satisfying*

$$0 \leq K_j \leq e^{P(j)}, \quad j \geq 0$$

where P is a polynomial of non-negative coefficients. Suppose that the sequence a_i satisfies

$$(6.3) \quad a_{j+1} \leq K_j(a_j^\kappa + a_j^{\kappa_1}), \quad a_j \geq 0, \quad j \geq 0.$$

Suppose that $\kappa_1 > \kappa > 1$. Set

$$C_j^* := \exp \left\{ \kappa^{-1}(\ln 2 + K(0)) + \sum_{m=1}^j \kappa^{-m}(\ln 2 + P(m)) \right\}, \quad 0 \leq j \leq \infty.$$

If $j \geq 0$ and $0 \leq C_j^ a_0 \leq 1$, then*

$$(6.4) \quad a_{j+1} \leq (C_j^* a_0)^{\kappa^j}.$$

In particular, if $0 \leq C_\infty^ a_0 \leq 1$, then $a_{j+1} \leq (C_\infty^* a_0)^{\kappa^j}$.*

Proof. Note that C_j^* increases with j and $C_j^* \geq 1$. Suppose that $0 \leq C_j^* a_0 \leq 1$. Then $a_0 \in [0, 1]$ and (6.3) imply that

$$a_1 \leq 2e^{P(0)} a_0^\kappa = (C_0^* a_0)^\kappa.$$

Suppose that (6.4) is verified for $j < m$. This implies that $a_m \leq 1$. By (6.3) we get

$$a_{m+1} \leq 2K_m a_m^\kappa \leq 2e^{P(m)} (C_{m-1}^* a_0)^{\kappa^m} = (C_m^* a_0)^{\kappa^m}. \quad \square$$

Proof of Proposition 1.2. Let us recall the proposition. We are in the special case of a family of complex structures on \mathbf{C}^n , defined near the origin by the adapted vector fields

$$(6.5) \quad Z_\alpha^\lambda = \partial_\alpha + A_{\alpha\beta}^\lambda(z) \partial_\beta, \quad 1 \leq \alpha \leq n, \quad A^\lambda(0) = 0, \quad |A^\lambda| < 1/C_0.$$

We want to find a family of transformations

$$F^\lambda: \hat{z} = z + f^\lambda(z), \quad f^\lambda(0) = 0, \quad |\partial_z^1 f| < 1/C_0$$

which transforms the complex structures into the standard one in \mathbf{C}^n .

Let us assume that $n \geq 2$. When $n = 1$ the following proof with simplifications remains valid; see Remark 6.4 following the proof.

The vector fields (6.5) are adapted so that the integrability condition has a simple form

$$(6.6) \quad [Z_\alpha^\lambda, Z_\beta^\lambda] = (Z_\alpha^\lambda A_{\beta\gamma}^\lambda - Z_\beta^\lambda A_{\alpha\gamma}^\lambda) \partial_\gamma = 0.$$

To simplify notation, let us drop λ in A^λ, f^λ , etc. We will use some abbreviations. If A, B are differential forms, $\langle A, B \rangle$ denotes a differential form of which the coefficients are finite sums of ab with a (resp. b) being a coefficient of A (resp. B). We will also further abbreviate $\langle A, B \rangle$ by AB sometimes. For $\{A^\lambda\}, \{B^\lambda\}$, the $\langle A, B \rangle$ denotes the family $\{\langle A^\lambda, B^\lambda \rangle\}$, while AB denotes the family $\{A^\lambda B^\lambda\}$.

Following [31] we form the following differential forms by using the coefficients of the adapted vector fields. With an abuse of notation, define

$$A_\beta = A_{\bar{\alpha}\beta} d\bar{z}_\alpha, \quad 1 \leq \beta \leq n.$$

Then the integrability condition (6.6) takes the form

$$(6.7) \quad \bar{\partial}A = \langle A, \partial A \rangle.$$

Let us compute the new vector fields after a change of coordinates by F . We have

$$F_* Z_{\bar{\alpha}} = (\delta_{\bar{\alpha}\bar{\beta}} + A_{\bar{\alpha}\gamma} \partial_\gamma \bar{f}_\beta) \partial_{\bar{z}_\beta} + (\partial_{\bar{\alpha}} f_\beta + A_{\bar{\alpha}\beta} + A_{\bar{\alpha}\gamma} \partial_\gamma f_\beta) \partial_{\hat{z}_\beta}.$$

The new adapted vector fields for $F_* S$ should still have the form

$$\hat{Z}_{\bar{\alpha}} = \partial_{\bar{\alpha}} + \hat{A}_{\bar{\alpha}\beta} \partial_\beta$$

where the new coefficients \hat{A} can be computed via

$$(6.8) \quad \begin{aligned} C_{\bar{\alpha}\bar{\beta}} \hat{Z}_{\bar{\beta}} &= F_* Z_{\bar{\alpha}}, \quad C_{\bar{\alpha}\bar{\beta}} \circ F = \delta_{\bar{\alpha}\bar{\beta}} + A_{\bar{\alpha}\gamma} \partial_\gamma \bar{f}_\beta, \\ (C_{\bar{\alpha}\gamma} \hat{A}_{\gamma\beta}) \circ F &= \partial_{\bar{\alpha}} f_\beta + A_{\bar{\alpha}\beta} + A_{\bar{\alpha}\gamma} \partial_\gamma f_\beta. \end{aligned}$$

Here A denotes (A_1, \dots, A_n) . Then F transforms $\{Z_{\bar{\alpha}}\}$ into the span of $\{\partial_{\bar{z}_\beta}\}$ if and only if $\{\hat{A}_\gamma\}$ are zero, i.e.

$$(6.9) \quad \bar{\partial}f + A + \langle A, \partial f \rangle = 0.$$

If \hat{A} is not zero, let us express it in terms of A . For $\hat{z} = F(z)$ and $\hat{A}_\beta = \hat{A}_{\bar{\alpha}\beta} d\bar{\hat{z}}_\alpha$, it is convenient to use

$$\hat{A}_\beta \circ F := \hat{A}_{\bar{\alpha}\beta} \circ F d\bar{z}_\alpha.$$

Then the new coefficients \hat{A} are given by

$$(6.10) \quad (C\hat{A}) \circ F = \bar{\partial}f + A + \langle A, \partial f \rangle.$$

While thinking of solving f for the non-linear equation (6.9), we use the homotopy formula to find an approximate solution. We then iterate to solve the equation eventually. For the approximate solution, we apply the homotopy formula as in [31]. On B_ρ^{2n} , we have the homotopy formula

$$\varphi = \bar{\partial}P_1\varphi + P_2\bar{\partial}\varphi.$$

To find approximate solutions to (6.9), we take

$$(6.11) \quad f_\beta = -P_1 A_\beta + (P_1 A_\beta)(0).$$

By the homotopy formula, where φ is a component of A , and the integrability condition (6.7) we write $\bar{\partial}f + A + \langle A, \partial f \rangle$ as $P_2 \langle A, \partial A \rangle + \langle A, \partial f \rangle$. Thus (6.10) becomes

$$(6.12) \quad (C\hat{A}) \circ F = \langle A, \partial f \rangle + P_2 \langle A, \partial A \rangle.$$

Without achieving $\hat{A} = 0$, we see that \hat{A} is written as products of two small functions. Hence, (6.11) is a good approximate solution. We now estimate \hat{A} .

Recall that

$$\begin{aligned}
|f|_{D;a,0} \odot |g|_{D';0,b} &:= |f|_{D;a,0} |g|_{D';0,[b]} + |f|_{D;[a],0} |g|_{D';0,b}, \\
Q_{D,D';a,b}^*(f, g) &:= |f|_{D;a,0} \odot |g|_{D';0,b} + |g|_{D';a,0} \odot |f|_{D;0,b}, \\
Q_{D,D';r,s}(f, g) &:= \sum_{k=0}^{\lfloor s \rfloor} Q_{D,D';r-j,j+\{s\}}^*(f, g), \quad [r] \geq [s], \\
(6.13) \quad \hat{Q}_{D,D';r,s}(f, g) &:= \|f\|_{D;r,s} |g|_{D';0,0} + |f|_{D;0,0} \|g\|_{D';r,s} + Q_{D,D';r,s}(f, g).
\end{aligned}$$

By Proposition A.3 we have the product rule

$$(6.14) \quad \left\| \left\{ \prod_{i=1}^m f_i^\lambda \right\} \right\|_{a,b} \lesssim \sum_{i \leq m} \|f_i\|_{a,b} \prod_{\ell \neq i} |f_\ell|_{0,0} + \sum_{i < j} Q_{a,b}(f_i, f_j) \prod_{\ell \neq i,j} |f_\ell|_{0,0},$$

$$(6.15) \quad \left\| \prod_{i=1}^{m-1} \phi_i(f_i^\lambda) \right\|_{a,b} \lesssim \sum_{i \leq m} \|f_i\|_{a,b} \prod_{\ell \neq i} |f_\ell|_{0,0} + \sum_{i < j} Q_{a,b}(f_i, f_j) \prod_{\ell \neq i,j} |f_\ell|_{0,0},$$

for $m \geq 2, a \geq b$, and $\phi_i(0) = 0$. Here $Q_{a,b}$ is defined by (3.1) and for the rest of the paper, by $A \lesssim B$ we mean that $A \leq CB$ for some constant C . We will simply use

$$Q_{a,b}(f, g) \lesssim |f|_{a,0} \odot |g|_{0,b} + |f|_{0,b} \odot |g|_{a,0}.$$

In particular, we have

$$(6.16) \quad Q_{a,b}(f, g) \lesssim |f|_{a,0} |g|_{0,b} + |f|_{0,b} |g|_{a,0}.$$

Let us derive estimates before we apply the rapid iteration. We first assume that $s \in \mathbb{N}$ and

$$\infty > r > s + 1, \quad \{r\} \neq 0.$$

Define $s_* = 0$ for $s = 0$, and $s_* = 1$ for $s \geq 1$. Define

$$\begin{aligned}
r_0 &= s_* + 1 + \{r\}, \quad r_1 = s + 1 + \{r\}, \\
\tilde{\rho}_i &= (1 - \theta)^i \rho, \quad i = 1, 2, 3.
\end{aligned}$$

We first use (5.12) to estimate the approximate solution f :

$$(6.17) \quad |f|_{\tilde{\rho}_1; m+1+\{r\}, s} \leq K(m+2) |A|_{\rho; m+\{r\}, s}.$$

Here and in the following $K(a)$ denotes a constant of the form

$$(6.18) \quad K(a) := K(a, \theta) = C_a \theta^{-a}.$$

For the rest of the proof, the norms of A and its derivatives are computed on B_ρ^{2n} , the norms of f for $F = I + f$ and its derivatives are computed on the domain $B_{\rho_1}^{2n}$, and the norms of g for $G := F^{-1} = I + g$ and its derivatives are computed on the domain $B_{\rho_2}^{2n}$. We will abbreviate $\|A\|_{a,b} = \|A\|_{\rho; r, s}$. Thus, all norms of A are on the domain B_ρ^{2n} .

Recall that $\hat{Q}_{r,s}(A, A)$ is defined by (6.13) via the \odot product. We define

$$Q_{r,s}^{**}(A, A) := \|A\|_{r_1, s} |A|_{r, 0} + \|A\|_{r_0, s_*} \|A\|_{r, s},$$

which is more suitable to use (6.17) which gains one derivative.

We want to estimate \hat{A} by its formula (6.12), which is rewritten as

$$(6.19) \quad (C\hat{A}) \circ F = A\partial f + P_2(A\partial A).$$

Let us first estimate various quadratic terms $Q_{r,s}$. By (6.14) or (A.30), we have

$$Q_{\rho,\rho;r-1,s}(A, \partial A) \lesssim \widehat{Q}_{r,s}(A, A) \lesssim Q_{r,s}^{**}(A, A).$$

From (6.17), we deduce two estimates

$$\begin{aligned} |f|_{\tilde{\rho}_1;1,s} &\leq |f|_{\tilde{\rho}_1;3/2,s} \leq K(2)\|A\|_{s+1,s}, \\ |f|_{\tilde{\rho}_1;r+1,0} &\leq K(r+2)|A|_{r,0}. \end{aligned}$$

We use the last three inequalities to estimate the following quadratic terms:

$$(6.20) \quad Q_{\rho,\tilde{\rho}_1;r,s}(A, \partial f) \lesssim |A|_{r,0}|f|_{\tilde{\rho}_1;1,s} + |A|_{0,s}|f|_{\tilde{\rho}_1;r+1,0} \leq K(r+2)Q_{r,s}^{**}(A, A),$$

$$(6.21) \quad Q_{\tilde{\rho}_1,\tilde{\rho}_1;r,s}(\partial f, \partial f) \lesssim |\partial f|_{\tilde{\rho}_1;r,0}|f|_{\tilde{\rho}_1;1,s} \leq K(2)K(r+2)Q_{r,s}^{**}(A, A).$$

Using the product rule (6.14) and (6.20), we verify

$$\begin{aligned} (6.22) \quad \|A\partial\bar{f}\|_{\tilde{\rho}_1;r,s} &\lesssim \|A\|_{r,s}|\partial\bar{f}|_{\tilde{\rho}_1;0,0} + |A|_{0,0}\|\partial\bar{f}\|_{\tilde{\rho}_1;r,s} + Q_{\rho,\tilde{\rho}_1;r,s}(A, \partial\bar{f}) \\ &\leq K(r+2)Q_{r,s}^{**}(A, A), \end{aligned}$$

$$(6.23) \quad \|A\partial A\|_{\tilde{\rho}_1;r-1,s} \lesssim Q_{r,s}^{**}(A, A).$$

Recall that $K(a) = K(a, \theta) = C_a\theta^{-a}$. Let

$$(6.24) \quad K_0(r_0, \theta) := K(r_0 + 2, \theta) \cdot \frac{C_{2n}}{\theta}.$$

To estimate the inverse of $F^\lambda = I + f^\lambda$, let us assume that

$$(6.25) \quad \|A\|_{\rho;r_0,s*} \leq K_0^{-1}(r_0, \theta).$$

Note that this implies that $Q_{r_0,s*}(A, A) \leq 1/C_n$. Then by (6.17) and (6.22) respectively, we have

$$\|f\|_{\tilde{\rho}_1;r_0+1,s*} \leq \theta/C_{2n}.$$

Thus we can use Proposition 3.2 or Lemma A.7 to estimate F and its inverse map. For $1/4 < \rho < 2$, we have

$$\begin{aligned} F^\lambda: B_{\tilde{\rho}_i}^{2n} &\rightarrow B_{\tilde{\rho}_{i-1}}^{2n}, \quad i = 1, 2; \\ G^\lambda: B_{\tilde{\rho}_2}^{2n} &\rightarrow B_{\tilde{\rho}_1}^{2n}, \quad F^\lambda \circ G^\lambda = I \quad \text{on } B_{\tilde{\rho}_2}^{2n}. \end{aligned}$$

By (6.19) we get $(C\hat{A}) \circ F = A\partial f + T(A\partial A)$. Then (6.22)-(6.23) yield

$$\begin{aligned} (6.26) \quad \|(C\hat{A}) \circ F\|_{\tilde{\rho}_1;r,s} &\leq K(r+2)(\|A\partial f\|_{\tilde{\rho}_1;r,s} + \|A\partial A\|_{\tilde{\rho}_1;r-1,s}) \\ &\lesssim K(r+2)Q_{r,s}^{**}(A, A). \end{aligned}$$

By (6.8), $C \circ F = I + A\partial\bar{f}$. By the product rule (6.15) and (6.20)-(6.22), we have

$$\begin{aligned} (6.27) \quad \|C^{-1} \circ F - I\|_{\tilde{\rho}_1;r,s} &\lesssim \|A\partial\bar{f}\|_{\tilde{\rho}_1;r,s} + Q_{r,s}(A\partial\bar{f}, A\partial\bar{f}) \\ &\leq K(2)K(r+2)Q_{r,s}^{**}(A, A). \end{aligned}$$

To estimate a new quadratic term, we express (6.26)-(6.27) for a special case:

$$\begin{aligned} (6.28) \quad |(C^{-1} \circ F - I)|_{\tilde{\rho}_1;r,0} &= |(I - A\partial\bar{f})^{-1}(A\partial\bar{f})|_{\tilde{\rho}_1;r,0} \\ &\leq K(2)K(r+2)|A|_{1,0}|A|_{r,0}, \end{aligned}$$

$$\begin{aligned} (6.29) \quad |(C\hat{A}) \circ F|_{\tilde{\rho}_1;r,0} &= |A\partial f + P_2(A\partial A)|_{\tilde{\rho}_1;r,0} \\ &\leq K(r+2)|A|_{1,0}|A|_{r,0}. \end{aligned}$$

Also, a direct estimation by (6.17) yields

$$(6.30) \quad \begin{aligned} |(C^{-1} \circ F - I)|_{\tilde{\rho}_1;0,s} &= |(I - A\partial\bar{f})^{-1}(A\partial\bar{f})|_{\tilde{\rho}_1;0,s} \\ &\leq K^2(2)|A|_{1,s}|A|_{1,0}, \end{aligned}$$

$$(6.31) \quad \begin{aligned} |(C\hat{A}) \circ F|_{\tilde{\rho}_1;0,s} &= |A\partial f + P_2(A\partial A)|_{\tilde{\rho}_1;0,s} \\ &\leq K(2)|A|_{1,s}|A|_{1,0}. \end{aligned}$$

Since $|A|_{r_0,s_*} \leq 1$ then from (6.28)-(6.31) we get a quadratic estimate

$$Q_{\tilde{\rho}_1,\tilde{\rho}_1;r,s}((C^{-1} \circ F - I), (C\hat{A}) \circ F) \leq K^3(2)K(r+2)Q_{r,s}^{**}(A, A).$$

By $\hat{A} \circ F = (C\hat{A}) \circ F + (C^{-1} \circ F - I)(C\hat{A}) \circ F$ and using the product rule (6.14), we get

$$(6.32) \quad \|\hat{A} \circ F\|_{\tilde{\rho}_1,\tilde{\rho}_1;r,s} \leq K^3(2)K(r+2)Q_{r,s}^{**}(A, A).$$

Set $K_1(r) = K^3(2)K(r+2)$. By (6.32), we have

$$\begin{aligned} \|\hat{A} \circ F\|_{1,s_*} &\leq K_1(r_0)Q_{r_0,s_*}^{**}(A, A) \leq 2K_1(r_0)\|A\|_{r_0,s_*}^2, \\ \|\hat{A} \circ F\|_{s+1,s} &\leq K_1(r_1)Q_{r_1,s}^{**}(A, A) \leq 2K_1(r_1)\|A\|_{r_1,s_*}\|A\|_{r_1,s}, \\ \|\hat{A} \circ F\|_{r,s_*} &\leq K_1(r)Q_{r,s_*}^{**}(A, A) \leq 2K_1(r_0)\|A\|_{r_0,s_*}\|A\|_{r,s_*}. \end{aligned}$$

By (6.17), we have $\|f\|_{\tilde{\rho}_1,r,s} \leq K(r)\|A\|_{r,s}$. Since $|f|_{\tilde{\rho}_1;1,0} \leq 1/C_N$ and $\|f\|_{\tilde{\rho}_1;0,s_*} \leq 1$, by (3.2) we have a general estimate

$$\begin{aligned} \|u \circ F^{-1}\|_{\tilde{\rho}_2;r,s} &\lesssim \{\|u\|_{\rho;r,s} + |u|_{\rho;1,s_*}\|f\|_{\rho;r,s} \\ &+ \|u\|_{s+1,s} \odot \|f\|_{r,s_*} + \|u\|_{r,s_*} \odot \|f\|_{s+1,s} + |u|_{\rho;1,0}\|f\|_{s+1,s} \odot \|f\|_{r,s_*}\}. \end{aligned}$$

Applying it to $u = A \circ F$, we obtain

$$(6.33) \quad \begin{aligned} |\hat{A}|_{\tilde{\rho}_2;r,s} &\lesssim \|\hat{A} \circ F\|_{\tilde{\rho}_1,\tilde{\rho}_1;r,s} + \|A\|_{\rho;r_0,s_*}\|A\|_{\tilde{\rho}_1;r,s} + 2\|A\|_{r_1,s_*}\|A\|_{r_1,s}\|A\|_{r,s_*} \\ &+ \|A\|_{r_1,s}\|A\|_{r,s_*}. \end{aligned}$$

Using (6.32), from (6.33) it follows

$$(6.34) \quad \|\hat{A}\|_{\tilde{\rho}_2;r,s} \leq C_r K_1(r) (Q_{\rho;r,s}^{**}(A, A) + \|A\|_{r_1,s_*}\|A\|_{r_1,s}\|A\|_{r,s_*}).$$

Case 1. $r > s+1$ and $r \notin \mathbf{N}$. We need to iterate the above construction and estimates to obtain a sequence of transformation F_j so that $\tilde{F}_i^\lambda := F_i^\lambda \circ \dots \circ F_0^\lambda$ converges to \tilde{F}_∞^λ , while $S_{i+1}^\lambda := (\tilde{F}_i)_* S_i^\lambda$, with S_0^λ being the original structure, converges to the standard complex structure on \mathbf{C}^n . More precisely, the coefficients A_i^λ of the adapted frame of Z_i^λ tend to 0 as $i \rightarrow \infty$. We apply Lemma 3.4 to nest the domains. Let us first defined a domain on which the sequence is well-defined. Set for $i = 0, 1, \dots$

$$\rho_i = \frac{1}{2} + \frac{1}{2(i+1)}, \quad \rho_{i+1} = (1 - \theta_i)^2 \rho_i.$$

Note that ρ_i decreases to $\rho_\infty = 1/2$. We have for $\rho_i/4 < \rho < 4\rho_i$, especially for $\rho_0/2 < \rho < 2\rho_0$

$$(6.35) \quad F_i^\lambda: B_{(1-\theta_i)\rho}^{2n} \rightarrow B_\rho^{2n}, \quad G_i^\lambda: B_{(1-\theta_i)^2\rho}^{2n} \rightarrow B_{(1-\theta_i)\rho}^{2n}$$

provided (6.25) holds for A_i , i.e.

$$(6.36) \quad \|A_i\|_{\rho_i,r_0,s_*} \leq K_0^{-1}(r_0, \theta_i).$$

By Lemma 3.4 in which $1 - \theta_k^* = (1 - \theta_k)^2$, we have $\tilde{F}_i^\lambda: B_{\rho_\infty/2} \rightarrow B_{\rho_\infty}$. Furthermore, in addition to (6.35), we express (6.34) in which $\hat{A} = A_{i+1}$ and $A = A_i$ in a simple form

$$(6.37) \quad \|A_{i+1}\|_{\rho_{i+1};r,s} \leq L_i(\|A_i\|_{\rho_i;r,s}^2 + \|A_i\|_{\rho_i;r,s}^3), \quad L_i := C_r K_1(r, \theta_i).$$

Therefore, for the sequence F_i to be defined, we need to achieve (6.36) for $i = 0, 1, 2, \dots$. Note that $K_0(r_0, \theta_i)$ and L_i have linear growth as $i \rightarrow \infty$. Thus

$$(6.38) \quad K_0(r_0, \theta_i) + L_i \leq e^{P(i)}$$

for some polynomial P of positive coefficients. Let constant C_∞^* be defined in Proposition 6.3 in which $\kappa = 2$ and $\kappa_1 = 3$. By Definition 6.1 and Lemma 6.2, there is $\hat{e}_0 > 0$ so that

$$(C_\infty^* \hat{e}_0)^{\kappa^i} \leq e^{-P(i)} \leq K_0^{-1}(r_0, \theta_i).$$

We may assume that $2C_\infty^* \hat{e}_0 \leq 1$. By Proposition 6.3 in which $\kappa = 2$ and (6.37)-(6.38), if

$$(6.39) \quad \|A_0\|_{\rho_0;r,s} \leq \hat{e}_0,$$

then

$$(6.40) \quad \|A_i\|_{\rho_i;r,s} \leq (C_\infty^* \hat{e}_0)^{\kappa^i}, \quad i \geq 1.$$

In particular, (6.36) hold for all i . On the other hand, (6.39) can be achieved by dilation and initial normalization in Lemma 2.1.

We now consider the convergence of the sequence $F_i \circ \dots \circ F_0$. The mapping F_i^λ transforms S_i^λ into S_{i+1}^λ . Thus the structure S_i^λ is defined on $B_{\rho_i}^{2n}$. Then $\tilde{F}_i^\lambda: B_{\rho_\infty/2}^{2n} \rightarrow B_{\rho_\infty}^{2n}$ transforms S_0^λ , restricted to $B_{\rho_\infty/2}^{2n}$, into S_{i+1}^λ . We have

$$\tilde{F}_i^\lambda - \tilde{F}_{i-1}^\lambda = f_i^\lambda \circ F_{i-1}^\lambda \circ \dots \circ F_0^\lambda.$$

By Proposition 3.3, we have

$$\begin{aligned} \|\{\tilde{F}_i^\lambda - \tilde{F}_{i-1}^\lambda\}\|_{\rho_\infty/2;r+1,s} &\leq C_r^{i+1} \left\{ \|f_i\|_{r+1,s} + \|f_i\|_{1,0} \sum_{k \leq j < i} \|f_k\|_{s+1,s} \|f_j\|_{r+1,s_*} \right. \\ &\quad \left. + \sum_{j < i} \|f_i\|_{1,s_*} \|f_j\|_{r+1,s} + \|f_i\|_{s+1,s} \|f_j\|_{r+1,s_*} + \|f_i\|_{r+1,s_*} \|f_j\|_{s+1,s} \right\} \\ &\leq C_r^{i+1} K_1(r) \left\{ \|A_i\|_{r,s} + \|A_i\|_{r_0,s_*} \sum_{k \leq j < i} \|A_k\|_{r_1,s} \|A_j\|_{r,s_*} \right. \\ &\quad \left. + \sum_{j < i} \|A_i\|_{r_0,s_*} \|A_j\|_{r,s} + \|A_i\|_{r_1,s} \|A_j\|_{r,s_*} + \|A_i\|_{r,s_*} \|A_j\|_{r_1,s} \right\}. \end{aligned}$$

Here the last inequality is obtained by (6.17). We already know the rapid convergence of $\|A_i\|_{\rho_i;r_0,s_*}$. They are bounded from above by a constant C_* . Then we simplify the above to obtain

$$(6.41) \quad \|\{\tilde{F}_i^\lambda - \tilde{F}_{i-1}^\lambda\}\|_{\rho_\infty/2;r+1,s} \leq \tilde{C}_r^{i+1} K^3(2) K(r+2) \|A_i\|_{\rho_i;r,s}.$$

The rapid decay of $\|A_i\|_{r,s}$ via (6.40) implies the convergence of \tilde{F}_i to \tilde{F}_∞ in $\mathcal{C}^{r+1,s}(B_{\rho_\infty/2}^{2n})$. Finally, for each λ , the limit of \tilde{F}_i^λ is a C^1 diffeomorphism defined in $B_{\rho_\infty/2}^{2n}$, because for the

Jacobian matrix $\partial_x \tilde{F}^i$ of $x \rightarrow \tilde{F}^i(x)$ with for $x \in B_{\rho_\infty/2}^{2n}$, its operator norm satisfies

$$\|\partial_y \tilde{F}_i^\lambda - I\| \leq |f_i|_{\rho_i;1,0} \prod_{m=0}^{i-1} (1 + |f_m|_{\rho_m;1,0}) \leq \|f_i\|_{\rho_i;r_0,s_*} \prod_{m=0}^{\infty} (1 + \|f_m\|_{\rho_m;r_0,s_*}) \leq 1/2.$$

This shows that the limit mapping \tilde{F}_∞ is a diffeomorphism.

Case 2. $r = \infty$. We first consider case where $r = \infty$ and s is finite. We first use (6.34) for $s = s_*$ and $r = r_0$ to get a rapid decay of $\|A_j\|_{r_0,s_*}$. Next, we use (6.34) in which $\hat{A} = A_{i+1}$ and $A = A_i$ a few times in a bootstrap argument. We first simplify (6.34) with $s = s_*$ and $r_1 = r_0 = 1 + s + \{r\}$ as

$$|A_{i+1}|_{\rho_{i+1};r',s_*} \leq C_{r'} K^3(2) K(r' + 2) \|A_i\|_{\rho_i;r_0,s_*} \|A_i\|_{\rho_i;r',s_*}$$

for any $r' > s + 1$ with $\{r'\} = \{r\}$. By Lemma 6.2 (ii), we get rapid decay of $\|A_i\|_{\rho_i;r',s_*}$. We can also simplify (6.34) as

$$(6.42) \quad \|A_{i+1}\|_{\rho_{i+1};r',s} \leq C_r K^3(2) K(r' + 2) \|A_i\|_{\rho_i;r',s_*} \|A_i\|_{\rho_i;r',s}.$$

By the rapid decay of $\|A_i\|_{\rho_i;r',s_*}$ and Lemma 6.2 (ii), we obtain the rapid convergence of $\|A_i\|_{\rho_i;r',s}$. By (6.41) again, we obtain the rapid convergence of $\|\tilde{F}_{i+1} - \tilde{F}_i\|_{\rho_\infty/2;r',s}$. This shows that \tilde{F}_∞ is in $\mathcal{C}^{\infty,s}$.

Next, we consider the case $s = \infty$. Applying the argument in Case 1 to $s = 1$, $r = 5/2$, we obtain rapid convergence of $\|\tilde{F}_{i+1} - \tilde{F}_i\|_{\rho_\infty/2;5/2,1}$. Next, we use (6.42) with $r' = s + 3/2$ for any positive integer s . By Lemma 6.2 (iii), we obtain the rapid convergence of $\|A_i\|_{\rho_i;s+3/2,s}$. By (6.41), we obtain the rapid convergence of $\|\tilde{F}_{i+1} - \tilde{F}_i\|_{\rho_\infty/2;s+3/2,s}$ for any positive integer s . This shows that \tilde{F}_∞ is in $\mathcal{C}^{\infty,\infty}$.

Case 3. $3 + s \leq r \in \mathbf{N}$. We first apply estimates in Case 1 to $r = r_1 = r_0 = s + \frac{5}{2}$. This gives us a rapid convergence of $\|A_i\|_{\rho_i;s+\frac{5}{2},s}$, $\|\tilde{F}_{i+1} - \tilde{F}_i\|_{\rho_\infty/2;s+\frac{7}{2},s}$. Fix $r - 1/2 < r' < r$. We want to show that $\|A_i\|_{\rho_i;r',s}$, $\|\tilde{F}_{i+1} - \tilde{F}_i\|_{\rho_\infty;r'+1,s}$ converge rapidly. By (6.34) we have

$$|A_{i+1}|_{\rho_{i+1};r',s} \leq C_{r'} K^3(2) K(r' + 2) \|A_i\|_{\rho_i;r_1,s} \|A_i\|_{\rho_i;r',s}.$$

Here $r_1 = s + 1 + \{r'\} < s + 2$. Since $\|A_i\|_{\rho_i;s+2,s}$ converges rapidly, then $\|A_{i+1}\|_{\rho_{i+1};r',s}$ also converges rapidly. We verify that $\|\tilde{F}_{i+1} - \tilde{F}_i\|_{\rho_\infty/2;r'+1,s}$ converges rapidly, by (6.41). Therefore, $\tilde{F}_\infty \in \mathcal{C}^{r'+1,s}$ for any $r' < r$. \square

Remark 6.4. Strictly speaking, the above proof assume that $n \geq 2$. When $n = 1$, the integrability condition (6.7) is vacuous. We can replace the homotopy formula by the Cauchy-Green operator

$$a(z) = \frac{-1}{\pi} \frac{\partial}{\partial \bar{z}} \int_{|\zeta| < \rho} \frac{a(\zeta)}{\zeta - z} d\xi d\eta, \quad \zeta = \xi + i\eta.$$

Thus the last term $P_2 \langle A, \partial A \rangle$ in (6.12), which is from the integrability condition, is removed. Therefore, the proof with possible simplifications is still valid. In one-dimensional case, one can also employ the Picard iteration as in Bers [1], Chern [5], and Bertrand-Gong-Rosay [3] (for a parametric version), except possibly for Case 3 where both r, s are integers. Of course, for the non-parametric case, it is a simple fact that any C^1 diffeomorphism that transforms a complex structure of class C^r into another one of the same class belongs to \mathcal{C}^{r-} when $r = 2, 3, \dots$. When $\mathcal{C}^{r,s}$ substitutes C^r , the latter is however not valid for the parametric complex structures.

We now prove the following by using an argument in Nirenberg [28] and Proposition 1.2.

Proposition 6.5. *Let $s \in \mathbf{N}$, $r \in (s+1, \infty] \setminus \mathbf{N}$. Let $\{S^\lambda\} \in \mathcal{C}^{r,s}(D)$ be a family of complex Frobenius structures defined by Theorem 1.3. There exists a family of $\{F^\lambda\} \in \mathcal{C}^{1,0}(U)$ transforming the structures into the standard complex Frobenius structure in $\mathbf{C}^n \times \mathbf{R}^M \times \mathbf{R}^L$. Furthermore, we have*

- (i) $\{F^\lambda\} \in \mathcal{C}^{r,s}(U)$ when $L = 0$.
- (ii) $\{F^\lambda\} \in \mathcal{C}^{[r]-1,s}(U)$ when $L > 0$ and $s = [r] - 1$.

Proof. We use an adapted frame

$$\begin{aligned} X_m^\lambda &= \partial'_m + \operatorname{Re}(B_{m\beta}^\lambda \partial_\beta) + b_{m\ell}^\lambda \partial_\ell'', & 1 \leq m \leq M, \\ Z_\alpha^\lambda &= \partial_\alpha + A_{\alpha\beta}^\lambda \partial_\beta + a_{\alpha\ell}^\lambda \partial_\ell'', & 1 \leq \alpha \leq n, \end{aligned}$$

with $X_m^\lambda = \partial'_m$ and $Z_\alpha^\lambda = \partial_\alpha$ at the origin.

(i) Suppose that $L = 0$. Then the integrability condition on S^λ implies that

$$[X_m^\lambda, X_i^\lambda] = 0, \quad [Z_\alpha^\lambda, Z_\beta^\lambda] = 0$$

(see (7.2) below), while the Levi-flatness condition (7.1) is vacuous. Restricted to $t = 0$, $\{Z_\alpha^\lambda\}$ defines a family of complex structures. By Proposition 1.2, we can find $\{F_0^\lambda\} \in \mathcal{C}^{r+1,s}(U')$ transforming the restricted structures into the standard complex structure in \mathbf{C}^n . Then $\{X_m^\lambda\}$ is still in $\mathcal{C}^{r,s}$. Applying the (real) Frobenius theorem (Proposition 4.3), we find a unique family $\{F^\lambda\} \in \mathcal{C}_{t,z;\lambda}^{r+1,r;s}(U)$ such that $F^\lambda = F_0^\lambda$ for $t = 0$ and F^λ transform $\{X_1^\lambda, \dots, X_M^\lambda\}$ into the standard structure in $\mathbf{C}^n \times \mathbf{R}^M$. As in [28], let us show that $\{Z^\lambda\}$ is already the standard complex structure in \mathbf{C}^n . Indeed, we know that

$$A_{\alpha\beta}^\lambda = 0, \quad \text{when } t = 0.$$

Now $[Z_\alpha, X_i]' = 0$ in (7.2) implies that $\partial_t A_{\alpha\beta}^\lambda(z, t) = 0$. Hence $A_{\alpha\beta} = 0$.

(ii) Suppose that $L > 0$. We reduce it to the previous case by using the Frobenius theorem. Recall that S^λ are Levi-flat. Applying Proposition 4.3, we find $F_0^\lambda(z, t, \xi)$ with $F_0^\lambda(0, 0, \xi) = \xi$ so that $X_j^\lambda F_0^\lambda = \bar{Z}_\alpha^\lambda F^\lambda = Z_\alpha^\lambda F_0^\lambda = 0$, while $\{F_0^\lambda\} \in \mathcal{C}_{(z,t),\xi;\lambda}^{r+1,r;s}$. Let \hat{F}_0^λ be defined by $(\hat{z}, \hat{t}, \hat{\xi}) = (z, t, F_0^\lambda(z, t, \xi))$. Then $(z, t, \xi) = (\hat{z}, \hat{t}, \hat{F}_0^\lambda(\hat{z}, \hat{t}, \hat{\xi}))$ with $\{\hat{F}_0^\lambda\} \in \mathcal{C}^{r,s}$. Now $\{(\hat{F}_0^\lambda)_* Z_\alpha^\lambda, (\hat{F}_0^\lambda)_* X_m^\lambda\}$, denoted by $\{\hat{Z}_\alpha^\lambda, \hat{X}_m^\lambda\}$, has the form

$$\begin{aligned} \hat{X}_m^\lambda &= \hat{\partial}'_m + \operatorname{Re}(B_{m\beta}^\lambda(\hat{z}, \hat{t}, \hat{F}_0^\lambda(\hat{z}, \hat{t}, \hat{\xi}))) \hat{\partial}_\beta, & 1 \leq m \leq M, \\ \hat{Z}_\alpha^\lambda &= \hat{\partial}_\alpha + A_{\alpha\beta}^\lambda(\hat{z}, \hat{t}, \hat{F}_0^\lambda(\hat{z}, \hat{t}, \hat{\xi})) \hat{\partial}_\beta, & 1 \leq \alpha \leq n. \end{aligned}$$

We drop all hats. Hence, $\{Z_\alpha^\lambda, X_m^\lambda\}$ is still in $\mathcal{C}^{r,s}$, but its coefficients depend on ξ . Since $[r] - 1 = s$, we treat ξ, λ as parameters, restrict Z_α^λ to $t = 0$ and find a diffeomorphism $\tilde{F}_1^\lambda: (z, \xi) \rightarrow (F_1^\lambda(z, \xi), \xi)$ such that for fixed (ξ, λ) the mapping transforms $\{Z_\alpha^\lambda\}$ into the standard complex structure in \mathbf{C}^n . Here $\{\tilde{F}_1^\lambda\} \in \mathcal{C}_{(z,t),\xi;\lambda}^{r+1,[r]-1;s}$. It also transforms X_m^λ to new vector fields, denoted by \hat{X}_m^λ . We have

$$\hat{X}_m = X_m t_i \partial_{t_i} + X_m \overline{(F_1^\lambda)_\alpha} \partial_{\bar{\alpha}} + X_m (F_1^\lambda)_\alpha \partial_\alpha.$$

Since X_m does not involve ∂_{ξ_ℓ} , then $\{\hat{X}_m^\lambda\} \in \mathcal{C}_{(z,t),\xi;\lambda}^{r,[r]-1;s}$. Again, we apply the real Frobenius theorem to $\{X_m^\lambda\}$, treating ξ, λ as parameters, to find

$$\tilde{F}_2^\lambda: (z, t, \xi) \rightarrow (F_2^\lambda(z, t, \xi), t, \xi), \quad F_2^\lambda(z, 0, \xi) = F_1^\lambda(z, \xi)$$

where $\{F_2^\lambda\} \in \mathcal{C}_{t,z,\xi;\lambda}^{r+1,r,[r]-1;s}$. We apply the argument from (i) to conclude that $\{\tilde{F}_2^\lambda\}$ normalizes the structures. \square

Roughly speaking Proposition 6.5 (ii) gives us a desired regularity result with loss of one derivative. One of main results of this paper is to deal with the loss of derivatives. We will also study the case when s is non integer.

All results in this paper indicate that for a family of Levi-flat CR structures, it is important to seek solutions that are less regularity in parameter λ . This is clearly showed by the examples in section 4. There is however an exception, namely, for a holomorphic family of complex structures, as shown by Nirenberg [28] for the \mathcal{C}^∞ case.

Definition 6.6. Let $r \in (0, \infty]$. Let D (resp. P, Q) be a domain in \mathbf{C}^n (resp. $\mathbf{C}^m, \mathbf{R}^k$). We say that $\{u^\lambda: \lambda \in P\}$ is a holomorphic family of \mathcal{C}^r (resp. \mathcal{C}^{r-}) functions u^λ in D if $(x, \lambda) \rightarrow u^\lambda(x)$ is continuous in $D \times P$, $u^\lambda \in \mathcal{C}^r(D)$ (resp. $\mathcal{C}^{r-}(D)$) for each λ , and $u^\lambda(x)$ is holomorphic in $\lambda \in P$ for each $x \in D$. We say that $\{v^{(\lambda,\mu)}: \lambda \in P, \mu \in Q\}$ is a family of \mathcal{C}^r functions in D that depend holomorphically in λ and analytically in μ , if there are a domain \tilde{Q} in \mathbf{C}^k with $\tilde{Q} \cap \mathbf{R}^k = Q$ and a holomorphic family $\{\tilde{v}^{\lambda,\mu}: \lambda \in P, \mu \in \tilde{Q}\}$ of \mathcal{C}^r (resp. \mathcal{C}^{r-}) functions $\tilde{v}^{\lambda,\mu}$ in D such that $\tilde{v}^{\lambda,\mu} = v^{\lambda,\mu}$ for $\mu \in Q$.

For the finite smoothness case, we have the following analogue of Nirenberg's theorem.

Corollary 6.7. Let D (resp. P, Q) be a domain in \mathbf{C}^n (resp. $\mathbf{C}^m, \mathbf{R}^k$). Let $S^{\lambda,\mu}$ be a family of complex structures in D defined by

$$Z_{\bar{\alpha}}^{\lambda,\mu} = a_{\bar{\alpha}\beta}^{\lambda,\mu} \frac{\partial}{\partial \bar{z}_\beta} + b_{\bar{\alpha}\beta}^{\lambda,\mu} \frac{\partial}{\partial z_\beta}, \quad 1 \leq \alpha \leq n$$

where $\{a_{\bar{\alpha}\beta}^{\lambda,\mu}\}, \{b_{\bar{\alpha}\beta}^{\lambda,\mu}\}$ are families of \mathcal{C}^r functions in D that depend holomorphic in $\lambda \in P$ and analytic in $\mu \in Q$. Then for each $(x_0, \lambda_0, \mu_0) \in D \times P \times Q$ there is a family of \mathcal{C}^{r+1} (resp. $\mathcal{C}^{(r+1)-}$) diffeomorphisms $F^{\lambda,\mu}$ in D_1 that depend holomorphically in $\lambda \in P_1$ and analytically in $\mu \in Q_1$ such that $F^{\lambda,\mu}$ transform $\{Z_1^{\lambda,\mu}, \dots, Z_{\bar{n}}^{\lambda,\mu}\}$ into the standard complex structure in \mathbf{C}^n , provided r is in $(1, \infty] \setminus \mathbf{N}$ (resp. $\{2, 3, \dots\}$). Here $D_1 \times P_1 \times Q_1$ is a neighborhood of (x_0, λ_0, μ_0) .

Proof. By the definition, we extend $a^{\lambda,\mu}, b^{\lambda,\mu}$ to a family of \mathcal{C}^r functions in D_0 that depend holomorphically in $(\lambda, \mu) \in P_0 \times Q_0$, where $D_0 \times P_0 \times Q_0$ is a neighborhood of (x_0, λ_0, μ_0) in $\mathbf{C}^n \times \mathbf{C}^m \times \mathbf{C}^k$. The extension of the coefficients allows us to extend $Z_1^{\lambda,\mu}, \dots, Z_{\bar{n}}^{\lambda,\mu}$ to a complex structure defined in $D_0 \times P_0 \times Q_0$, when we include the vector fields

$$\frac{\partial}{\partial \bar{\lambda}_i}, \quad \frac{\partial}{\partial \bar{\mu}_j}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq k.$$

By Cauchy's formula, we verify that $(z, \lambda, \mu) \rightarrow (a^{\lambda,\mu}(x), b^{\lambda,\mu}(x))$ is \mathcal{C}^r in (z, λ, μ) . Assume first that $r > 1$ is not an integer. By Webster's result on Newlander-Nirenberg theorem, we can find a \mathcal{C}^{r+1} diffeomorphism $(z, \lambda, \mu) \rightarrow (f_1, \dots, f_{n+m+k})$ such that $Z_{\bar{\alpha}}^{\lambda,\mu} f_\ell = 0$ while f_j are holomorphic in λ, μ . Near (z_0, λ_0, μ_0) , by a linear combination we may assume that $z \rightarrow (f_1(z, \lambda_0, \mu_0), \dots, f_n(z, \lambda_0, \mu_0))$ is a diffeomorphism near z_0 . Let $F_\ell^{\lambda,\mu} = f_\ell^{\lambda,\mu}$ for $1 \leq \ell \leq n$. Then $F^{\lambda,\mu}$ has the desired property. When $r = 2, 3, \dots$, we use Proposition 1.2. The proof is complete. \square

7. APPROXIMATE SOLUTIONS VIA TWO HOMOTOPY FORMULAE

In this section, we prove the results for the general case. We will first set up a procedure for our problem to apply a homotopy formula for the normalization of the complex Frobenius stricture. We will need two known homotopy formulae. One of them due to Treves [30] is for the complex differential \mathcal{D} associated the flat CR structure in $\mathbf{C}^n \times \mathbf{R}^m$. Another is the Poincare lemma for $d^0 + \bar{\partial} + \partial$.

We will finish the section by illustrating how the homotopy formula is used more effectively. See also Gong-Webster [11] for the case of CR embedding, which improves Webster's original use of the homotopy formula for $\bar{\partial}_b$.

On \mathbf{C}^n , set $\partial_{\bar{\alpha}} = \partial_{\bar{z}^\alpha}$ and $\partial_\alpha = \partial_{z^\alpha}$. On \mathbf{R}^m , set $\partial'_m = \partial_{t_m}$. On \mathbf{R}^L , set $\partial''_\ell = \partial_{\xi_\ell}$. Let $N = 2n + M + L$. To normalize S near a point p , we may assume that $p = 0 \in D \subset \mathbf{R}^N$ and require that all changes of coordinates fix the origin. By Lemma 2.1 we first choose linear coordinates of \mathbf{R}^N so that near 0, S is spanned by the adapted frame

$$\begin{aligned} X_m &= \partial'_m + 2 \operatorname{Re}(B_{m\beta} \partial_\beta) + b_{m\ell} \partial''_\ell, \quad 1 \leq m \leq M, \\ Z_{\bar{\alpha}} &= \partial_{\bar{\alpha}} + A_{\bar{\alpha}\beta} \partial_\beta + a_{\bar{\alpha}\ell} \partial''_\ell, \quad 1 \leq \alpha \leq n. \end{aligned}$$

Here $A_{\bar{\alpha}\beta}, B_{m\ell}, a_{\bar{\alpha}\ell}$ are complex-valued functions and $b_{m\ell}$ are real-valued. They all vanish at the origin. Define $\mathcal{A} = (A, B, a, b)$ and assume that

$$|(A, B, a, b)| < 1/C_0.$$

Then $Z_{\bar{\alpha}}, Z_\alpha := \overline{Z_{\bar{\alpha}}}$, and X_m are pointwise \mathbf{C} -linearly independent. Set

$$\overline{A_{\bar{\alpha}\beta}} := \overline{A_{\bar{\alpha}\beta}}, \quad \overline{B_{m\beta}} := \overline{B_{m\beta}}, \quad \overline{a_{\bar{\alpha}\ell}} := \overline{a_{\bar{\alpha}\ell}}.$$

The Lie brackets of the adapted frame are

$$\begin{aligned} [Z_{\bar{\alpha}}, Z_{\bar{\beta}}] &= (Z_{\bar{\alpha}} A_{\bar{\beta}\gamma} - Z_{\bar{\beta}} A_{\bar{\alpha}\gamma}) \partial_\gamma + (Z_{\bar{\alpha}} a_{\bar{\beta}\ell} - Z_{\bar{\beta}} a_{\bar{\alpha}\ell}) \partial''_\ell, \\ [X_m, X_i] &= 2 \operatorname{Re}\{(X_m B_{i\gamma} - X_i B_{m\gamma}) \partial_\gamma\} + (X_m b_{i\ell} - X_i b_{m\ell}) \partial''_\ell, \\ [Z_{\bar{\alpha}}, X_m] &= [Z_{\bar{\alpha}}, X_m]' \mod Z_{\bar{\gamma}}, \end{aligned}$$

with

$$\begin{aligned} [Z_{\bar{\alpha}}, X_m]' &:= (Z_{\bar{\alpha}} B_{m\gamma} - X_m A_{\bar{\alpha}\gamma} - A_{\bar{\beta}\gamma} Z_{\bar{\alpha}} \overline{B_{m\beta}}) \partial_\gamma \\ &\quad + (Z_{\bar{\alpha}} b_{m\ell} - X_m a_{\bar{\alpha}\ell} - a_{\bar{\beta}\ell} Z_{\bar{\alpha}} \overline{B_{m\beta}}) \partial''_\ell. \end{aligned}$$

To find equations for $S + \bar{S}$ to be Levi-flat, we compute Levi-forms

$$[Z_\alpha, Z_{\bar{\beta}}] = Z_\alpha A_{\bar{\beta}\gamma} \partial_\gamma - Z_{\bar{\beta}} \overline{A_{\bar{\alpha}\gamma}} \partial_{\bar{\gamma}} + (Z_\alpha a_{\bar{\beta}\ell} - Z_{\bar{\beta}} \overline{a_{\bar{\alpha}\ell}}) \partial''_\ell.$$

Define the matrix

$$R := (I - A\bar{A})^{-1} - I, \quad R = (R_{\bar{\alpha}\beta}), \quad \bar{R} = (\bar{R}_{\bar{\alpha}\bar{\beta}}), \quad U := I + R.$$

Let us substitute ∂_α and $\partial_{\bar{\alpha}}$ by using

$$\begin{aligned} \partial_{\bar{\alpha}} &= Z_{\bar{\alpha}} - A_{\bar{\alpha}\beta} Z_{\bar{\beta}} - a_{\bar{\alpha}\ell} \partial''_\ell + R_{\bar{\alpha}\bar{\beta}} Z_{\bar{\beta}} - R_{\bar{\alpha}\bar{\beta}} A_{\bar{\beta}\gamma} Z_\gamma \\ &\quad - (R_{\bar{\alpha}\bar{\beta}} a_{\bar{\beta}\ell} - U_{\bar{\alpha}\bar{\beta}} A_{\bar{\beta}\mu} \overline{a_{\mu\ell}}) \partial''_\ell. \end{aligned}$$

Write $[X_\alpha, X_{\bar{\beta}}] = L_{\ell; \alpha \bar{\beta}} \partial_\ell'' \pmod{(Z_{\bar{\gamma}}, Z_\mu)}$ with Levi forms $(L_{\ell; \alpha \bar{\beta}})_{1 \leq \alpha, \beta \leq n}$ being defined by

$$(7.1) \quad \begin{aligned} L_{\ell; \alpha \bar{\beta}} := & Z_\alpha a_{\bar{\beta} \ell} - Z_{\bar{\beta}} \bar{a}_{\alpha \ell} + Z_\alpha A_{\bar{\beta} \gamma} \bar{U}_{\bar{\gamma} \delta} (\bar{A}_{\delta \nu} a_{\bar{\nu} \ell} - \bar{a}_{\delta \ell}) \\ & - Z_{\bar{\beta}} \bar{A}_{\alpha \gamma} U_{\bar{\gamma} \delta} (A_{\delta \nu} \bar{a}_{\bar{\nu} \ell} - a_{\delta \ell}), \quad 1 \leq \alpha, \beta \leq n. \end{aligned}$$

Then the integrability conditions are equivalent to

$$(7.2) \quad [Z_{\bar{\alpha}}, Z_{\bar{\beta}}] = 0, \quad [X_m, X_i] = 0, \quad [Z_{\bar{\alpha}}, X_m]' = 0, \quad L_{\ell; \alpha \bar{\beta}} = 0.$$

We now introduce some differential forms by using the adapted frame. On \mathbf{C}^n and \mathbf{R}^m we introduce the following $(0, 1)$ and 1 forms

$$(7.3) \quad \begin{aligned} A_\beta &= A_{\bar{\alpha} \beta} d\bar{z}_\alpha, \quad a_\ell = a_{\bar{\alpha} \ell} d\bar{z}_\alpha, \\ B_\beta &= B_{m\beta} dt_m, \quad b_\ell = b_{m\ell} dt_m, \\ \mathcal{B}_\beta &= A_\beta + B_\beta, \quad \mathbf{b}_\ell = a_\ell + \bar{a}_\ell + b_\ell. \end{aligned}$$

For a function f in \mathbf{R}^N , we denote by $\bar{\partial}f = \partial_{\bar{\alpha}} f d\bar{z}_\alpha$ and $d_t f = \partial'_m f dt_m$. Recall that $\partial_{z, \xi} u$ stands for first-order derivatives of u in $\mathbf{C}^n \times \mathbf{R}^L$.

The conditions (7.2) are equivalent to

$$(7.4) \quad \bar{\partial}A = \langle (A, a), \partial_{z, \xi} A \rangle, \quad \bar{\partial}a = \bar{\partial}\bar{a} = \langle (A, a), \partial_{z, \xi} a \rangle,$$

$$(7.5) \quad d_t B = \langle (B, b), \partial_{z, \xi} B \rangle, \quad d_t b = \langle (B, b), \partial_{z, \xi} b \rangle,$$

$$(7.6) \quad \bar{\partial}B + d_t A = \langle (A, a), \partial_{z, \xi} B \rangle + \langle (B, b), \partial_{z, \xi} A \rangle + \langle A, \partial_{z, \xi} B \rangle + \langle (A, a), \partial_{z, \xi} B \rangle,$$

$$(7.7) \quad \bar{\partial}b + d_t a = \langle (A, a), \partial_{z, \xi} b \rangle + \langle (B, b), \partial_{z, \xi} a \rangle + \langle a, \partial_{z, \xi} B \rangle + \langle a, \partial_{z, \xi} B \rangle,$$

$$(7.8) \quad \partial a + \bar{\partial}a = 2 \operatorname{Re} \{ \langle A, \partial_{z, \xi} a \rangle + \langle P(R(A), A, a)(\bar{A}, \bar{a}), \partial_{z, \xi} A \rangle \}.$$

Recall that $d = d_t + \partial_z + \bar{\partial}_z$ and $\mathcal{D} = d_t + \bar{\partial}_z$. Define

$$(7.9) \quad \mathcal{A} := (\mathcal{B}, \mathbf{b}), \quad \mathbf{d}\mathcal{A} := (\mathcal{D}\mathcal{B}, d\mathbf{b}).$$

Thus we can rewrite the above conditions as

$$(7.10) \quad \mathbf{d}\mathcal{A} = \langle \psi(\mathcal{A})\mathcal{A}, \partial_{z, \xi} \mathcal{A} \rangle.$$

Here $\psi(\mathcal{A})$ are power series in the coefficients of $\mathcal{A}, \bar{\mathcal{A}}$, which are convergent for $|\mathcal{A}| < 1/C$ for some constant $C > 1$.

We consider a transformation

$$H: \hat{z} = z + f(z, t, \xi), \quad \hat{t} = t, \quad \hat{\xi} = \xi + g(z, t, \xi)$$

with $f(0) = 0, g(0) = 0, |\partial f(0)| < 1$ and $|\partial g(0)| < 1$. Define $\hat{\partial}_\alpha = \partial_{z_\alpha}, \hat{\partial}'_m = \partial_{t_m}$, and $\hat{\partial}''_\ell = \partial_{\xi_\ell}$, etc. Then

$$\begin{aligned} H_* Z_{\bar{\alpha}} &= (\delta_{\bar{\alpha} \beta} + (A_{\bar{\alpha} \gamma} \partial_\gamma + a_{\bar{\alpha} \ell} \partial_\ell'') \bar{f}_\beta) \hat{\partial}_{\bar{\beta}} + (\partial_{\bar{\alpha}} f_\beta + A_{\bar{\alpha} \beta} + (A_{\bar{\alpha} \gamma} \partial_\gamma + a_{\bar{\alpha} \ell} \partial_\ell'') f_\beta) \hat{\partial}_{\bar{\beta}} \\ &\quad + (\partial_{\bar{\alpha}} g_\ell + a_{\bar{\alpha} \ell} + (A_{\bar{\alpha} \gamma} \partial_\gamma + a_{\bar{\alpha} m} \partial_m'') g_\ell) \hat{\partial}_\ell'', \\ H_* X_m &= \hat{\partial}'_m + 2 \operatorname{Re}((\partial_m f_\beta + B_{m\beta} + (2 \operatorname{Re}(B_{m\gamma} \partial_\gamma) + b_{mj} \partial_j'') f_\beta) \hat{\partial}_\beta) \\ &\quad + (\partial'_m g_\ell + b_{m\ell} + (2 \operatorname{Re}(B_{m\gamma} \partial_\gamma) + b_{mj} \partial_j'') g_\ell) \hat{\partial}_\ell''. \end{aligned}$$

Assume that

$$f(0) = 0, \quad g(0) = 0, \quad |\partial f| + |\partial g| < 1/C_0.$$

Then the new adapted frame for H_*S is $\widehat{Z}_{\bar{\alpha}} = \widehat{\partial}_{\bar{\alpha}} + \widehat{A}_{\bar{\alpha}\beta}\widehat{\partial}_{\beta} + \widehat{a}_{\bar{\alpha}\ell}\widehat{\partial}_{\ell}''$ and $\widehat{X}'_m = \widehat{\partial}'_m + 2\operatorname{Re}(\widehat{B}_{m\beta}\widehat{\partial}_{\beta}) + \widehat{b}_{m\ell}\widehat{\partial}_{\ell}''$. Their coefficients are determined as follows. We have

$$\widehat{X}_m = H_*X_m, \quad C_{\bar{\alpha}\bar{\beta}}\widehat{Z}_{\bar{\beta}} = H_*Z_{\bar{\alpha}}$$

with $C_{\bar{\alpha}\bar{\beta}} \circ H = \delta_{\bar{\alpha}\bar{\beta}} + (A_{\bar{\alpha}\gamma}\partial_{\gamma} + a_{\bar{\alpha}\ell}\partial_{\ell}'')\bar{f}_{\beta}$. Also

$$\begin{aligned} C_{\bar{\alpha}\bar{\beta}} \circ H &= \delta_{\bar{\alpha}\bar{\beta}} + (A_{\bar{\alpha}\gamma}\partial_{\gamma} + a_{\bar{\alpha}\ell}\partial_{\ell}'')\bar{f}_{\beta}, \\ (C_{\bar{\alpha}\bar{\gamma}}\widehat{A}_{\bar{\gamma}\beta}) \circ H &= \partial_{\bar{\alpha}}f_{\beta} + A_{\bar{\alpha}\beta} + (A_{\bar{\alpha}\gamma}\partial_{\gamma} + a_{\bar{\alpha}\ell}\partial_{\ell}'')f_{\beta}, \\ (C_{\bar{\alpha}\bar{\gamma}}\widehat{a}_{\bar{\gamma}\ell}) \circ H &= \partial_{\bar{\alpha}}g_{\ell} + a_{\bar{\alpha}\ell} + (A_{\bar{\alpha}\gamma}\partial_{\gamma} + a_{\bar{\alpha}m}\partial_m'')g_{\ell}, \\ \widehat{B}_{m\beta} \circ H &= \partial'_m f_{\beta} + B_{m\beta} + (2\operatorname{Re}(B_{m\gamma}\partial_{\gamma}) + b_{m\ell}\partial_{\ell}'')f_{\beta}, \\ \widehat{b}_{m\ell} \circ H &= \partial'_m g_{\ell} + b_{m\ell} + (2\operatorname{Re}(B_{m\gamma}\partial_{\gamma}) + b_{mj}\partial_j'')g_{\ell}. \end{aligned}$$

By an abuse of notation, define

$$\begin{aligned} \widehat{A}_{\beta} \circ H &= \widehat{A}_{\bar{\alpha}\beta} \circ Hd\bar{z}_{\alpha} + \widehat{B}_{m\beta} \circ Hdt_m, \\ \widehat{a}_{\ell} \circ H &= 2\operatorname{Re}\{\widehat{a}_{\alpha\ell} \circ Hdz_{\alpha}\} + \widehat{b}_{m\ell} \circ Hdt_m. \end{aligned}$$

Thus, H transforms $\{Z_{\bar{\gamma}}, X_i\}$ into the span of $\{\widehat{\partial}_{\bar{\delta}}, \widehat{\partial}_k\}$ if and only if \widehat{A}_{β} , \widehat{B}_{β} , \widehat{a}_{ℓ} , and \widehat{b}_{ℓ} are zero, i.e.

$$(7.11) \quad \mathcal{D}f + \mathcal{B} + \langle (\mathcal{B}, \mathfrak{b}), \partial_{z,t}f \rangle = 0,$$

$$(7.12) \quad dg + \mathfrak{b} + \langle (\mathcal{B}, \mathfrak{b}), \partial_{z,t}g \rangle = 0.$$

When $\widehat{\mathcal{A}} = (\mathcal{B}, \mathfrak{b})$ is non-zero, we have, for \mathbf{d} defined by (7.9),

$$(7.13) \quad (C\widehat{\mathcal{A}}) \circ H = \mathbf{d}h + \mathcal{A} + \langle \mathcal{A}, \partial_{z,t}h \rangle, \quad h := (f, g), \quad C\widehat{\mathcal{A}} = (C\widehat{\mathcal{B}}, \widehat{\mathfrak{b}}).$$

As in Webster [31], we apply homotopy formulae. In our case, we apply the (approximate) Poincaré lemma and the Koppelman-Leray formula for forms of degree 1 and $(0, 1)$ to find approximate solutions to (7.11)-(7.12). We will then use the compatibility conditions (7.4)-(7.8) and two homotopy formulae to verify that the solutions f and g are indeed good approximate solutions to (7.11)-(7.12).

Set $D_{\rho} = B_{\rho}^{2n} \times B_{\rho}^M \times B_{\rho}^L$. Recall that $\mathcal{D} = d^0 + \bar{\partial}$ is defined in $\mathbf{C}^n \times \mathbf{R}^M$ and d is the standard real differential in $\mathbf{C}^n \times \mathbf{R}^M$. We have homotopy formulae

$$\begin{aligned} \varphi_q &= \mathcal{D}T_{B_{\rho}^{2n} \times B_{\rho}^M} \varphi_q + T_{B_{\rho}^{2n} \times B_{\rho}^M} \mathcal{D}\varphi_{(0,q)}, \quad q > 0, \\ \psi_q &= dR_{B_{\rho}^{2n} \times B_{\rho}^M} \psi_q + R_{B_{\rho}^{2n} \times B_{\rho}^M} d\psi_q, \quad q > 0. \end{aligned}$$

where T_{ρ} is defined by (5.6), and R_{ρ} is defined by (5.1) in which \mathbf{R}^M, B_{ρ} are replaced by $\mathbf{C}^n \times \mathbf{R}^M, B_{\rho}^{2n} \times B_{\rho}^M$ respectively. Define $\mathcal{T}_{\rho}(\mathcal{B}, \mathfrak{b}) := (T_{B_{\rho} \times B'_{\rho}}\mathcal{B}, R_{B_{\rho} \times B'_{\rho}}\mathfrak{b})$. Thus we have the homotopy formula

$$\mathcal{A} = \mathbf{d}\mathcal{T}_{\rho}\mathcal{A} + \mathcal{T}_{\rho}\mathbf{d}\mathcal{A}.$$

To find approximate solutions to (7.11)-(7.12), we take

$$(7.14) \quad h = -\mathcal{T}_{\rho}\mathcal{A} + (\mathcal{T}_{\rho}\mathcal{A})(0).$$

Note that g is real-valued, as required. By (7.10), formally, we obtain $|\widehat{\mathcal{A}}| \lesssim |\mathcal{A}|^2$. However, the argument cannot be repeated infinitely many times as \widehat{A} in (7.13) is less smooth than \mathcal{A} because of no gain in derivative from the estimate of $\mathcal{T}_{\rho}\mathcal{A}$.

Instead, we use a smoothing operator S_τ in (z, t, ξ) variables on D_ρ , and take

$$(7.15) \quad H(z, t, \xi) = (z + f_\tau(z, t, \xi), t, \xi + g_\tau(z, t, \xi)),$$

$$(7.16) \quad h_\tau = (f_\tau, g_\tau) := -S_\tau \mathcal{T}_\rho \mathcal{A} + S_\tau \mathcal{T}_\rho \mathcal{A}(0).$$

On shrinking domains D_{ρ_1} , we have

$$[S_\tau, \mathbf{d}] = 0.$$

As in [31], we apply the homotopy formula, a second time, to write $\mathbf{d}\mathcal{T}_\rho \mathcal{A}$ as $\mathcal{A} - \mathcal{T}_\rho \mathbf{d}\mathcal{A}$. We obtain

$$\begin{aligned} \mathcal{A} + \mathbf{d}h_\tau &= \mathcal{A} - \mathbf{d}S_\tau \mathcal{T}_\rho \mathcal{A} = \mathcal{A} - S_\tau \mathbf{d}\mathcal{T}_\rho \mathcal{A} \\ &= (I - S_\tau)\mathcal{A} + S_\tau \mathcal{T}_\rho \mathbf{d}\mathcal{A}. \end{aligned}$$

Now equation (7.13) for $\widehat{\mathcal{A}}$ is changed to

$$(7.17) \quad (\mathcal{C}\widehat{\mathcal{A}}) \circ H = \mathcal{A} + \mathbf{d}h_\tau + \langle \mathcal{A}, \partial h_\tau \rangle = I_1 + I_2 + I_3$$

where $\mathcal{C}\widehat{\mathcal{A}} = (C\widehat{\mathcal{B}}, \widehat{\mathbf{b}})$ and

$$(7.18) \quad I_1 = (I - S_\tau)\mathcal{A}, \quad I_2 = S_\tau \mathcal{T}_\rho \mathbf{d}\mathcal{A}, \quad I_3 = \langle \mathcal{A}, \partial S_\tau \mathcal{T}_\rho \mathcal{A} \rangle.$$

We introduce

$$(7.19) \quad C \circ H - I = I_4, \quad I_4 = \langle \mathcal{A}, \partial S_\tau \mathcal{A} \rangle.$$

Here $I_{4,\bar{\alpha}\beta} = -(A_{\bar{\alpha}\gamma}\partial_\gamma + B_{\bar{\alpha}\ell}\partial_\ell'')S_\tau(P_\rho \mathcal{A})_\beta$. We also have

$$(7.20) \quad \mathbf{d}\mathcal{A} = \langle \psi(\mathcal{A})\mathcal{A}, \partial \mathcal{A} \rangle.$$

Notice that all I_i have the smoothing operator S_τ . And I_4 will be estimated as I_3 as they have the same form. The I_1 will dictate the regularity result when we apply the iteration method.

8. THE PROOF FOR A FAMILY OF COMPLEX FROBENIUS STRUCTURES

In this section, we prove the general version of our theorem. Recall that the proofs in section 6 rely on gaining one full derivative in the Koppelman-Leray homotopy formula. Such a gain does not exist in the general case. Thus in an iteration procedure, we need to apply the Nash-Moser smoothing methods. By using smoothing operators, we will also be able to get rid of the extra one derivative required in the non-parametric variables, i.e., we will consider complex Frobenius structures of class $\mathcal{C}^{r,s}$ requiring $r \geq s$ and $r > 1$, instead of $r > s + 1$.

Besides the more elaborated use of homotopy formula, we will also need to use the interpolation of Hölder norms to estimate the norms for new complex Frobenius structures and to obtain rapid convergence in higher order derivatives via the rapid convergence in lower order derivatives. Note that the interpolation is avoided in the rapid KAM arguments in section.

We first recall estimates for smoothing operator

$$(8.1) \quad \|S_\tau f\|_{D,b} \leq C_{b-a} \tau^{a-b} \|f\|_{D',a}, \quad 0 \leq b - a < \infty;$$

$$(8.2) \quad \|f - S_\tau f\|_{D,a} \leq C_{a-b} \tau^{b-a} \|f\|_{D',b}, \quad 0 \leq b - a \leq m_0.$$

Here S_τ depends on $m_0 \in \mathbf{N}$. See Moser [23], where the proof of above inequalities are for integers a, b, m . The general case can be derived by interpolation in Hölder norms for domains of the cone property, as used in [11].

For the Hölder spaces with parameter, we will use the same smoothing operator. The approximation is subtle and we derive full details. Fix a positive integer L . There exists a smooth function φ in \mathbf{R}^{N_0} with compact support in the unit ball such that

$$(8.3) \quad \int P(x)\psi(x) dx = P(0),$$

where P is a polynomial of degree at most L ; see Moser [23]. The smoothing operator for functions f with compact support is

$$S_\tau f = \psi_\tau * f, \quad \psi_\tau = \tau^{-n} \psi(\tau x), \quad \tau > 0.$$

It will be convenient to use Seeley extension operator [29] to extend $\{f^\lambda: 0 \leq t \leq 1\}$ to a larger family defined for λ in a larger interval, say $\tilde{I} = [-1, 2]$. Let us recall the extension. Seeley [29] showed that there are numerical sequences $\{a_k\}, \{b_k\}$ so that (i) $b_k < 0$ and $b_k \rightarrow -\infty$, $(-1)^k a_k > 0$, (ii) $\sum |a_k| \cdot |b_k|^n < \infty$ for $n = 0, 1, 2, \dots$, (iii) $\sum a_k (b_k)^n = 1$ for $n = 0, 1, 2, \dots$. When $f^\lambda = 0$ for $t \geq 1/2$, we can define the extension

$$(8.4) \quad (E^s f)(y) = \sum_{k=0}^{\infty} a_k \phi(b_k s) f^{b_k s}(y), \quad s \leq 0.$$

Here ϕ is a \mathcal{C}^∞ function satisfying $\phi(\lambda) = 1$ for $\lambda < 1$ and $\phi(\lambda) = 0$ for $\lambda > 2$. For a differential form f , we define Ef by extending coefficients of f via E . Using a partition of unity on $[0, 1]$, we obtain an extension $E: C^0([0, 1]) \rightarrow C_0^0((-1, 2))$. Applying the extension to the parameter λ , we define function $\{E^\lambda f\} \in \mathcal{C}^{0,0}(\overline{D})$ for $\lambda \in [-1, 2]$. Furthermore,

$$|\{E^\lambda f\}|_{r,s} \leq C_{r,s} |f|_{r,s},$$

where $C_{r,s}$ depends on r and \mathcal{C}^r coordinate charts mapping ∂D into the half-space, and D is independent of λ . In our applications, D is a ball in a Euclidean space of which the radius is bounded between two fixed positive numbers. Therefore, $C_{r,s}$ depends only on r, s .

We will use two smoothing operators. The first smoothing operator is S_τ , applied to all variables (y, λ) . This is suitable for functions in $\mathcal{C}^{r,s}$ for $r = s$. When $r > s$, we will use a *partial* smoothing operator S_τ , by smoothing in variables $y \in D$. Let us exam the effects of partial smoothing operator, which is still denoted by S_τ , using (8.1)-(8.4). For real numbers a, b and s , define

$$d_s(a, b) = \tilde{d}_s(a, b) = b - a, \quad s \in \mathbf{N};$$

$$d_s(a, b) = \min(b - a, [b] - [a]), \quad \tilde{d}_s(a, b) = \max(b - a, [b] - [a]), \quad s \notin \mathbf{N}.$$

When s is not an integer, $d_s(a, b) > 0$ if and only if $[b] > [a]$. Furthermore, if $\{r\} \geq \{s\}$, then

$$d_s(a, b) > 1 \Leftrightarrow b > a + 1 \text{ and } s \in \mathbf{N}, \text{ or } [b] \geq [a] + 2.$$

The following is a basic property of partial smoothing operator.

Proposition 8.1. *Let $D \subset D'$ be domains in \mathbf{R}^N . Suppose that D has the cone property. Let $0 < \tau < \min\{1, \text{dist}(D, \partial D')\}$. Then*

$$(8.5) \quad |S_\tau f|_{D;b,s} \leq C_{b-a} \tau^{-\tilde{d}_s(a,b)} |f|_{D';a,s}, \quad 0 \leq a \leq b < \infty,$$

$$(8.6) \quad |f - S_\tau f|_{D;a,s} \leq C_{b-a} \tau^{d_s(a,b)} |f|_{D';b,s}, \quad 0 \leq a \leq b < a + m_0,$$

$$(8.7) \quad \|S_\tau f\|_{D;b,s} \leq C_{b-a} \tau^{-\tilde{d}_s(a,b)} \|f\|_{D';a,s}, \quad 0 \leq s \leq a \leq b < \infty,$$

$$(8.8) \quad \|f - S_\tau f\|_{D;a,s} \leq C_{b-a} \tau^{d_s(a,b)} \|f\|_{D';b,s}, \quad 0 \leq s \leq a \leq b < a + m_0.$$

Here C_a depends on two constants C_1^*, C_2^* in the cone property of D and bounded from above by an upper bound of a .

Proof. It suffices to verify the inequalities for $0 \leq a < 1$ and $0 \leq s < 1$, since

$$d_s(a, b) = d_{s-n}(a, b) = d_s(a - m, b - m), \quad \tilde{d}_s(a, b) = \tilde{d}_{s-n}(a, b) = \tilde{d}_s(a - m, b - m),$$

for integers m, n . Let ∂^k be a derivative in $y \in D'$ of order k . Let ∂_λ be the partial derivative in parameter λ . We have $\partial^k S_\tau f^\lambda = S_\tau \partial^k f^\lambda$ and $\partial_\lambda S_\tau f^\lambda = S_\tau \partial_\lambda f^\lambda$ on shrinking domain $D' \times [0, 1]$.

(i) Recall from (8.3) that there is a smooth function χ with support in $|y| < 1/2$ so that

$$(8.9) \quad \int \chi(y) dy = 1, \quad \int \chi(y) y^I dy = 0, \quad 0 < |I| \leq m_0 + 1.$$

Let $\chi_\tau(y) = \tau^{-n} \chi(\tau^{-1}y)$. We have $S_\tau f^\lambda = f^\lambda * \chi_\tau$. Thus $\|\{S_\tau f^\lambda\}\|_{D;a,s} \leq C \|f\|_{D';a,s}$. We have $\partial^k S_\tau f^\lambda = f^\lambda * \partial^k \chi_\tau$. Since

$$\tau < \tau_0 := \min\{1, \text{dist}(D, \partial D')\},$$

we have

$$|\partial^k(\chi_\tau(y))| \leq C_k \tau^{-k-n} \sum_{|I| \leq k} |(\partial^I \chi)(\tau^{-1}y)|.$$

This shows that $|S_\tau f|_{D;a+k,s} \leq C_k \tau^{-k} |f|_{D';a,s}$, which also gives us the first inequality when a, b are an integer. For $k - 1 < b < k$, we write $b = a + \theta k$ and obtain

$$|S_\tau f|_{D;b,[s]} \leq C_b |S_\tau f|_{D';a,[s]}^{1-\theta} |S_\tau f|_{D';a+k,[s]}^\theta \leq C'_b \tau^{a-b} |f|_{D';a,[s]}.$$

When $\{s\} > 0$, we have for $[b] = [a] + \theta([b] - [a])$

$$\left| \frac{\partial_{\lambda'}^{[s]} S_\tau f^{\lambda'} - \partial_\lambda^{[s]} S_\tau f^\lambda}{|\lambda' - \lambda|^{\{s\}}} \right|_{D;[b]} \leq C_b |S_\tau f|_{D';[a],s}^{1-\theta} |S_\tau f|_{D';[a]+k,s}^\theta \leq C'_b \tau^{[a]-[b]} |f|_{D';a,s}.$$

(ii) Let $k \leq b < k + 1$. Let $y \in D$. Let $P_k^\lambda(y, y')$ be the Taylor polynomial of f^λ of degree k about y so that

$$|\partial_\lambda^{[s]} f^\lambda(y + y') - \partial_\lambda^{[s]} P_k^\lambda(y, y')| \leq C_b |f|_{D';b,[s]} |y'|^b, \quad |y'| < \tau_0.$$

By (8.9), $\int P_k^\lambda(y, y') \chi(y') dy' = f^\lambda(y)$. We have

$$\begin{aligned} S_\tau f^\lambda(y) &= f^\lambda(y) + \int \{f^\lambda(y - \tau y') - P_k^\lambda(y, -\tau y')\} \chi(y') dy' \\ &= f^\lambda(y) + \mathcal{I}^\lambda(y). \end{aligned}$$

We have $|f^\lambda(y + \tau y') - P_k^\lambda(y, \tau y')| \leq C_b |f|_{D'; b, [s]} \|\tau y'\|^b$ for $|\tau y'| < \tau_0$. Therefore

$$|\mathcal{I}^\lambda(y)| \leq C_b \tau^b |f|_{D'; b, [s]} \int_{|y| < 1/2} |y|^b |\chi(y)| dy \leq C'_b \tau^b |f|_{D'; b, [s]}.$$

This completes the proof of the second inequality when $a = 0$. For $0 < a < 1$, we have

$$\begin{aligned} \left| \partial_\lambda^{[s]} f^\lambda - \partial_\lambda^{[s]} S_\tau f^\lambda \right|_{D; a} &\leq C_b \left| \partial_\lambda^{[s]} f^\lambda - \partial_\lambda^{[s]} S_\tau f^\lambda \right|_{D; 0}^{\frac{b-a}{b}} \left| \partial_\lambda^{[s]} f^\lambda - \partial_\lambda^{[s]} S_\tau f^\lambda \right|_{D; b}^{\frac{a}{b}} \\ &\leq C'_b (\tau^b |f|_{D'; b, [s]})^{\frac{b-a}{b}} \|f\|_{D'; b, [s]}^{\frac{a}{b}}. \end{aligned}$$

We can also estimate the Hölder ratio in t as in (i). This completes the proof. \square

Remark 8.2. The cone property is used only for interpolation. Thus Proposition 8.1 hold for integral a, b without the cone property of D . We will use (8.5) and (8.8) only for $b - a = 1$, in which case $\tilde{d}_s(a, b) = 1$.

The following is the main ingredient in the Nash-Moser iteration procedure. Another ingredient is the interpolation inequality. As in Gong-Webster [11], it is useful to adjust the parameters for iteration.

Proposition 8.3. *Let κ_0, κ, d be numbers bigger than 1. Suppose that*

$$(\kappa - 1)d > 1.$$

Let P, P_0 be polynomials of non-negative coefficients. Let K_j be a sequence of numbers satisfying

$$0 \leq K_j \leq e^{P(j)}$$

for $j \geq 0$. Suppose that a_j, L_j are two sequences of nonnegative numbers satisfying

$$(8.10) \quad a_{j+1} \leq K_j (\tau^d L_j + a_j^{\kappa_0} L_j + \tau^{-1} a_j^\kappa), \quad j \geq 0,$$

$$(8.11) \quad L_{j+1} \leq K_j (1 + \tau^{-1} a_j) L_j, \quad j \geq 0,$$

for $0 < \tau \leq e^{-P_0(j)}$. Let $\hat{L}_0 \geq 0$. Suppose that κ_, d_* satisfy*

$$d > d_* \kappa_*, \quad d_* > 1, \quad \kappa_0 > \kappa_* \geq 1, \quad d_*(\kappa - \kappa_*) > 1.$$

Let $\tau_{j+1} = \tau_j^{d_}$ for $j \geq 0$. There exist positive numbers $\hat{a}_0, \hat{\tau}_0$ in $(0, 1)$, which depend only on $\hat{L}_0, \kappa, \kappa_0, \kappa_*, d, d_*, P$ and P_0 , so that if $0 \leq L_0 \leq \hat{L}_0$, $0 \leq a_0 \leq \hat{a}_0$ and $0 < \tau_0 \leq \hat{\tau}_0$ then*

$$(8.12) \quad a_j \leq \tau_j^{\kappa_*}, \quad L_{j+1} \leq L_0 e^{P_1(j)},$$

for $j \geq 0$, where $P_1(j) = j + P(1) + \dots + P(j)$.

Remark 8.4. Condition (8.10) ensures the existence of κ_*, d_* that satisfy (8.12).

Proof. Suppose that $0 < \tau_0 < 1$. By the definition of τ_j , $\tau_j = \tau_0^{d_*^j}$. By Lemma 6.2, we have

$$\tau_j^{\kappa_* - 1} \leq 1, \quad \tau_j \leq e^{-P_0(j)},$$

when $\tau_0 \leq \hat{\tau}_0$. Here $\hat{\tau}_0$ depends on P_0 . Suppose that we can verify the first inequality in (8.12). From (8.11) we also have

$$L_{j+1} \leq 2K_j L_j \leq e^{1+P(j)} L_j \leq e^{P_1(j)} L_0.$$

This shows the second inequality in (8.12). For the first inequality, we get $a_0 \leq \tau_0^{\kappa_*}$ by choosing \hat{a}_0 depending on $\hat{\tau}_0$ and κ_* . By (8.10) we deduce

$$\tau_{j+1}^{-\kappa_*} a_{j+1} \leq \tau_j^{-\kappa_* d_*} \{L_0 e^{P(j)+P_1(j)} (\tau_j^d + \tau_j^{\kappa_0 d_*}) + e^{P(j)} \tau_j^{\kappa d_* - 1}\}.$$

For the three components of τ_j , we have $d - \kappa_* d_* > 0$, $(\kappa_0 - \kappa_*) d_* > 0$, and $\kappa d_* - 1 - \kappa_* d_* = d_*(\kappa - \kappa_*) - 1 > 0$. This shows that $\tau_j^{\kappa - \kappa_* d_*}$, $\tau_j^{(\kappa_0 - \kappa_*) d_*}$, and $\tau_j^{(\kappa - \kappa_*) d_* - 1}$ converge rapidly. By (6.2), we have

$$\tau_j^{-\kappa_* d_*} \{L_0 e^{P(j)+P_1(j)} (\tau_j^d + \tau_j^{\kappa_0 d_*}) + e^{P(j)} \tau_j^{\kappa d_* - 1}\} \leq 1,$$

which is achieved by choosing $\hat{\tau}_0$ that depends on $\kappa_0, \kappa, \kappa_*, d, d_*, \hat{L}_0, P, \kappa_0$, as well as P_0 indicated early. Note that \hat{a}_0 also depends on $\hat{\tau}_0$. We have proved that $\tau_{j+1}^{-\kappa_*} a_{j+1} \leq 1$. \square

Let us restate Theorem 1.3 here. We also organize the statements in the order of the proof.

Theorem 8.5. *Let $\{S^\lambda\} \in \mathcal{C}^{r,s}$ be as in Theorem 1.3. Then we can find $\{F^\lambda\} \in \mathcal{C}^{1,0}$ satisfies the assertion in Theorem 1.3. Furthermore, we have*

- (i) $\{F^\lambda\} \in \mathcal{C}^{r-,s}(U)$, provided $r > s + 2 \in \mathbf{N}$.
- (ii) $\{F^\lambda\} \in \mathcal{C}^{\infty,s}(U)$, provided $r = \infty$, and $s \in [1, \infty]$.
- (iii) $\{F^\lambda\} \in \mathcal{C}^{r-,s-}(U)$, provided $r \geq s + 3$ and $\{r\} \geq \{s\} > 0$.
- (iv) $\{F^\lambda\} \in \mathcal{C}^{r-,r-}(U)$, provided $r = s > 1$.
- (v) $\{F^\lambda\} \in \mathcal{C}^{r-,0}(U)$, provided $r > 1$.

Proof. For the first three parts we will not apply the Nash-Moser smoothing operator to the parameter λ . The integrability condition (8.14) already involves a derivative in $x \in \mathbf{R}^N$. Therefore, for the iteration method to work we requires that $\{S^\lambda\} \in \mathcal{C}^{r,s}$ with $r \geq s + 1$.

(i) Let us recall the approximation for a change of coordinates described at the end of section 7. Recall that \mathcal{A} is defined by (7.9) and (7.3) for the adapted frame of S . Applying the transformation H defined by $H = I + h_\tau$ in (7.15)-(7.16), we obtain a new complex Frobenius structure $\hat{S} = H_* S$ of which the corresponding $\hat{\mathcal{A}}$ has the form

$$(8.13) \quad (\mathcal{C}\hat{\mathcal{A}}) \circ H = I_1 + I_2 + I_3.$$

Recall from (7.18) that $\mathcal{C}\hat{\mathcal{A}} = (C\hat{\mathcal{B}}, \hat{\mathbf{b}})$ and

$$I_1 = (I - S_\tau)\mathcal{A}, \quad I_2 = S_\tau \mathcal{T}_\rho \mathbf{d}\mathcal{A}, \quad I_3 = \langle \mathcal{A}, \partial S_\tau \mathcal{T}_\rho \mathcal{A} \rangle.$$

By (7.19), the matrix C satisfies

$$I_4 := C \circ H - I = \langle \mathcal{A}, \partial S_\tau \mathcal{T}_\rho \mathcal{A} \rangle.$$

By (7.20), the integrability condition has the form

$$(8.14) \quad \mathbf{d}\mathcal{A} = \langle \psi(\mathcal{A})\mathcal{A}, \partial\mathcal{A} \rangle.$$

The estimates for I_4 and I_3 will be identical. By (8.13), we obtain

$$\hat{\mathcal{A}} \circ H = (I + I_4)^{-1} (I_1 + I_2 + I_3).$$

We first assume that

$$\infty > r \geq s + 1, \quad \{r\} \geq \{s\}.$$

In what follows, we assume that

$$(8.15) \quad \|\mathcal{A}\|_{\rho;s+1,s} \leq K_0^{-1}(s, \theta)\tau.$$

Here $K_0(r_0, \theta) := 2K(r_0 + 2, \theta) \cdot \frac{C_N}{\theta}$, which is analogous to (6.24). The factor τ is needed in reflecting the need of applying Nash-Moser smoothing operator. We need to derive estimates on shrinking domains. Set

$$\tilde{\rho}_i = (1 - \theta)^i \rho, \quad i = 1, 2, 3.$$

To apply estimate for S_τ , we assume that

$$(8.16) \quad 0 < \tau < \rho\theta/C_0.$$

By estimate (5.14) on $D_{\tilde{\rho}_1}$ for the homotopy formulae and estimate on $D_{\tilde{\rho}_2}$ for S_τ , we obtain

$$(8.17) \quad |S_\tau \mathcal{T}_\rho \mathcal{A}|_{\tilde{\rho}_2; \ell, a} \leq K(\ell + 1) |\mathcal{A}|_{\rho; \ell, a}.$$

Therefore, we have

$$\|I_1\|_{\tilde{\rho}_1; 0, 0} \lesssim \|\mathcal{A}\|_{\rho; 0, 0}, \quad \|I_i\|_{\tilde{\rho}_2; 0, 0} \leq K(1) \|A\|_{\rho; 0, 0} \|\mathcal{A}\|_{\rho; 1, 0}, \quad 2 \leq i \leq 4.$$

Note that $|I_4|_{\tilde{\rho}_2; 0, 0} \leq 1/(2N)$ and $|I_i|_{\tilde{\rho}_2; 0, 0} \leq 1$. Applying the product rule in Lemma 3.1, we obtain

$$\begin{aligned} \|\hat{\mathcal{A}} \circ H\|_{\tilde{\rho}_2; r, s} &= \|(I + I_4)^{-1}(I_1 + I_2 + I_3)\|_{\tilde{\rho}_2; r, s} \\ &\lesssim \sum_{i=1}^3 \|I_i\|_{\tilde{\rho}_2; r, s} + \|I_i\|_{\tilde{\rho}_2; b, s} |I_4|_{\tilde{\rho}_2; 0, 0} + \|I_i\|_{\tilde{\rho}_2; 0, 0} \|I_4\|_{\tilde{\rho}_2; r, s} \\ &\quad + \sum_{i=1}^3 \|I_i\|_{\tilde{\rho}_2; 0, 0} Q_{\tilde{\rho}_2, \tilde{\rho}_2; r, s}(I_4, I_4) + Q_{\tilde{\rho}_2, \tilde{\rho}_2; r, s}(I_i, I_4) \\ &\lesssim \sum_{i=1}^4 \|I_i\|_{\tilde{\rho}_2; r, s} + Q_{\tilde{\rho}_2, \tilde{\rho}_2; r, s}(I_i, I_4). \end{aligned}$$

Recall that $h = -S_\tau \mathcal{T}_\rho \mathcal{A} + S_\tau \mathcal{T}_\rho \mathcal{A}(0)$. By (8.17),

$$(8.18) \quad |h|_{\tilde{\rho}_2; \ell, a} \leq 2K(\ell + 1) |\mathcal{A}|_{\rho; \ell, a}.$$

Then (8.15) and (8.18) imply

$$(8.19) \quad \|h\|_{\tilde{\rho}_2; s+1, s} \leq \theta/C_N.$$

To simplify notation, let $\tilde{\mathcal{A}} = \hat{\mathcal{A}} \circ H$. Write $\hat{\mathcal{A}} = \tilde{\mathcal{A}} \circ H^{-1}$. By (8.19) we can apply (3.2), and by (8.18) we get from (3.2)

$$(8.20) \quad \begin{aligned} \|\hat{\mathcal{A}}\|_{\tilde{\rho}_3; r, s} &\leq C_r \left\{ \|\tilde{\mathcal{A}}\|_{\tilde{\rho}_2; r, s} + |\tilde{\mathcal{A}}|_{\tilde{\rho}_2; 1, s_*} \|\mathcal{A}\|_{\tilde{\rho}_2; r, s} + \|\tilde{\mathcal{A}}\|_{\tilde{\rho}_2; s+1, s} \|\mathcal{A}\|_{\tilde{\rho}_2; r, s_*} \right. \\ &\quad \left. + \|\tilde{\mathcal{A}}\|_{\tilde{\rho}_2; 1, 0} \|\mathcal{A}\|_{\tilde{\rho}_2; s+1, s} \odot \|\mathcal{A}\|_{\tilde{\rho}_2; r, s_*} + \|\tilde{\mathcal{A}}\|_{\tilde{\rho}_2; r, s_*} \odot \|\mathcal{A}\|_{\tilde{\rho}_2; s+1, s} \right\}. \end{aligned}$$

We need to estimate $\tilde{\mathcal{A}} = (I + I_4)^{-1}(I_1 + I_2 + I_3)$ by the product rule. By the properties of the smoothing operator, we get

$$(8.21) \quad \|I_1\|_{\tilde{\rho}_1; \ell, s} \lesssim \tau^{d_s(b, \ell)} \|\mathcal{A}\|_{b, s}, \quad s \leq \ell \leq b < \ell + m_0,$$

$$(8.22) \quad \|I_1\|_{\tilde{\rho}_1; b, s} \lesssim \|\mathcal{A}\|_{b, s}, \quad s \leq b < \infty.$$

Here and in what follows, we write $\|\mathcal{A}\|_{a, b} = \|\mathcal{A}\|_{\rho; a, b}$ and $|\mathcal{A}|_{a, b} = |\mathcal{A}|_{\rho; a, b}$ for simplicity. By (8.5) and (8.7), we have

$$|I_4|_{\tilde{\rho}_2; \ell, a} = |\langle \mathcal{A}, \partial S_\tau \mathcal{T}_\rho \mathcal{A} \rangle|_{\tilde{\rho}_2; \ell, a} \leq K(\ell) \tau^{-1} (|\mathcal{A}|_{\ell, a} |\mathcal{A}|_{0, 0} + |\mathcal{A}|_{\ell, 0} |\mathcal{A}|_{0, a}).$$

Recall that $\widehat{Q}_{b,s}(\mathcal{A}, \mathcal{A}) = \|A\|_{b,s} \|\mathcal{A}\|_{0,0} + Q_{b,s}(\mathcal{A}, \mathcal{A})$. Since I_3 has the same form as I_4 , then we have verified for $i = 3, 4$

$$\begin{aligned} \|I_i\|_{\tilde{\rho}_2;r,s} &\leq K(r+2)\tau^{-1}(\|\mathcal{A}\|_{r,s}|\mathcal{A}|_{0,0} + |\mathcal{A}|_{r,0} \odot |\mathcal{A}|_{0,s}) \\ \|I_i\|_{\tilde{\rho}_2;s+1,s} &\leq K(s+2)\|\mathcal{A}\|_{s+1,s}. \end{aligned}$$

Here the second inequality follows from the first and $|\mathcal{A}|_{s+1,s} \leq K_0^{-1}(1)\tau$ by (8.15).

The estimates for I_2 need the integrality condition $d\mathcal{A} = \langle \psi(\mathcal{A})\mathcal{A}, \partial\mathcal{A} \rangle$. By (6.15) the latter gives us

$$\begin{aligned} (8.23) \quad |I_2|_{\tilde{\rho}_2;\ell,a} &= |S_\tau \mathcal{T}_\rho d\mathcal{A}|_{\tilde{\rho}_2;\ell,a} \leq K(\ell)\tau^{-1}|\mathcal{T}_\rho d\mathcal{A}|_{\tilde{\rho}_1;\ell-1,a} \\ &\leq K(\ell)\tau^{-1}\left\{ |\mathcal{A}|_{\ell-1,a}|\mathcal{A}|_{1,0} + |\mathcal{A}|_{0,0}|\mathcal{A}|_{\ell,a} \right. \\ &\quad \left. + |\mathcal{A}|_{\ell-1,0} \odot |\mathcal{A}|_{1,a} + |\mathcal{A}|_{0,a} \odot |\mathcal{A}|_{\ell,0} \right\} \\ &\leq K(\ell)\tau^{-1}(|\mathcal{A}|_{\ell,a}|\mathcal{A}|_{0,0} + |\mathcal{A}|_{\ell,0} \odot |\mathcal{A}|_{0,a}), \quad \ell \in [1, \infty), \\ |I_2|_{\tilde{\rho}_2;0,s} &\leq K(1)(|\mathcal{A}|_{1,s}|\mathcal{A}|_{0,0} + |\mathcal{A}|_{0,s}|\mathcal{A}|_{1,0}). \end{aligned}$$

In particular, with $\|\mathcal{A}\|_{s+1,s} \leq \tau$ we have

$$\|I_2\|_{\tilde{\rho}_2;s+1,s} \leq K(s+1)\|\mathcal{A}\|_{s+1,s}.$$

Since $[r] \geq [s] + 1$, from (8.23) we conclude

$$\|I_2\|_{\tilde{\rho}_2;r,s} \leq K(r)\tau^{-1}(\|\mathcal{A}\|_{r,s}|\mathcal{A}|_{0,0} + |\mathcal{A}|_{r,0} \odot |\mathcal{A}|_{0,s}).$$

Since $|I_i|_{\tilde{\rho}_2;0} \leq 1/2$ and $\tilde{\mathcal{A}} = (I + I_4)^{-1}(I_1 + I_2 + I_3)$, we have

$$\begin{aligned} (8.24) \quad \|\tilde{\mathcal{A}}\|_{\tilde{\rho}_2;r,s} &\lesssim \sum_{i=1}^4 \|I_i\|_{\tilde{\rho}_2;r,s} + \sum_{i=1}^4 (\|I_i\|_{\tilde{\rho}_2;0,s} \|I_4\|_{\tilde{\rho}_2;r,0} + \|I_4\|_{\tilde{\rho}_2;0,s} \|I_i\|_{\tilde{\rho}_2;r,0}) \\ &\lesssim \|I_1\|_{\tilde{\rho}_2;r,s} + K(r+1)\tau^{-1}(\|\mathcal{A}\|_{r,s}|\mathcal{A}|_{0,0} + |\mathcal{A}|_{r,0} \odot |\mathcal{A}|_{0,s}). \end{aligned}$$

Using $|I_i|_{\tilde{\rho}_2;s+1,s} \leq K(s+2)\|\mathcal{A}\|_{s+1,s}$ and (8.22), we obtain its first consequence

$$(8.25) \quad \|\tilde{\mathcal{A}}\|_{\tilde{\rho}_2;r,s} \leq K(r+1)\|\mathcal{A}\|_{r,s}.$$

By (8.20), we immediately obtain

$$\|\widehat{\mathcal{A}}\|_{\tilde{\rho}_3;r,s} \leq K(r+1)\|\mathcal{A}\|_{r,s}.$$

We can also take $b = s+1$ in (8.20) and use (8.25) to simplify all quadratic terms on the right-hand side of (8.20). We conclude

$$\|\mathcal{A}\|_{\tilde{\rho}_3;s+1,s} \leq C_r \|\tilde{\mathcal{A}}\|_{\rho_1;s+1,s} + K(s+2)\|\mathcal{A}\|_{s+1,s}^2.$$

To improve it, we use (8.24) again by taking $b = s+1$. We now use (8.21) in which $\ell = s+1$ to achieve

$$(8.26) \quad \|\widehat{\mathcal{A}}\|_{\tilde{\rho}_3;s+1,s} \leq C_r \tau^{d_s(r,s+1)} \|\mathcal{A}\|_{r,s} + K(s+2)\tau^{-1}\|\mathcal{A}\|_{s+1,s}^2.$$

We have derived necessary estimates. We will use (8.26) to get rapid convergence in $\mathcal{C}^{s+1,s}$ norms, while (8.25) is for controlling the high order derivative in linear growth. This allows us to achieve rapid convergence for intermediate derivatives via interpolation.

Assume now that $d := d_s(r, s+1) > 1$, and s is a positive integer.

We briefly recall part of the proof of Proposition 1.2 for nesting domains. We need to find a sequence of transformations F_j so that $\tilde{F}_i^\lambda := F_i^\lambda \circ \dots \circ F_0^\lambda$ converges to \tilde{F}_∞^λ , while $S_{i+1}^\lambda := (\tilde{F}_i^\lambda)_* S_i^\lambda$, with S_0^λ being the original structure, converges to the standard complex Frobenius structure in \mathbf{R}^N . Set for $i = 0, 1, \dots$

$$\rho_i = \frac{1}{2} + \frac{1}{2(i+1)}, \quad \rho_{i+1} = (1 - \theta_i)^3 \rho_i.$$

Thus, we take $1 - \theta_k^* = (1 - \theta_k)^3$ in Lemma 3.4. Let $\rho_\infty = \lim_{i \rightarrow \infty} \rho_i$. Recall that

$$D_\rho = B_\rho^{2n} \times B_\rho^M \times B_\rho^L.$$

We have for $\rho_0/2 < \rho < 2\rho_0$

$$(8.27) \quad H_i: D_{(1-\theta_i)\rho} \rightarrow D_\rho, \quad H_i^{-1}: D_{(1-\theta_i)^2\rho} \rightarrow D_{(1-\theta_i)\rho}$$

provided τ_i satisfies (8.16), and (8.15) holds for \mathcal{A}_i , i.e.

$$(8.28) \quad \tau_i \leq \rho_i \theta_i / C_0,$$

$$(8.29) \quad \|\mathcal{A}_i\|_{\rho_i; s+1, s} \leq K_0^{-1}(s+2, \theta_i) \tau_i, \quad K_0(s+2, \theta_i) := \frac{4K(s+2, \theta_i)}{(\rho_i \theta_i)^{s+2}}.$$

Then (8.25) and (8.26) take the form

$$(8.30) \quad \|\mathcal{A}_{i+1}\|_{\rho_{i+1}; s+1, s} \leq K(s+2, \theta_i) (\tau_i^d \|\mathcal{A}_i\|_{\rho_i; r, s} + \tau_i^{-1} \|\mathcal{A}_i\|_{\rho_i; s+1, s}^2),$$

$$(8.31) \quad \|\mathcal{A}_{i+1}\|_{\rho_{i+1}; r, s} \leq K(r+1, \theta_i) \|\mathcal{A}_i\|_{\rho_i; r, s}.$$

We want to apply Proposition 8.3 in which $a_i = \|\mathcal{A}_i\|_{\rho_i; s+1, s}$ and $L_i = \|\mathcal{A}_i\|_{\rho_i; r, s}$. Obviously, there are polynomials P, P_0 satisfying

$$K(r, \theta_i) \leq e^{P(i)}, \quad K_0(s+2, \theta_i) \leq e^{P_0(i)}.$$

Applying Proposition 8.3 to (8.30)-(8.31), we find $d_* > 1$, $0 < \tau_0 = \hat{\tau}_0 < 1$ such that for $\tau_{j+1} = \tau_j^{d_*}$, we have

$$(8.32) \quad \|\mathcal{A}_i\|_{\rho_i; s+1, s} \leq \tau_i^{\kappa_*} \leq K_0^{-1}(s+2, \theta_i) \tau_i$$

for some $\kappa_* > 1$, provided we have

$$(8.33) \quad \|\mathcal{A}_0\|_{\rho_0; s+1, s} \leq \hat{a}_0.$$

Here \hat{a}_0 and $\hat{\tau}_0$ are fixed now and they depend only on the polynomials P, P_0 , and d . We have achieved (8.33) in (2.4) via an initial normalization and a dilation. Therefore, we have achieved (8.32) for all i . By (8.18), (8.31) and Lemma 6.2, we know that $\|h_i\|_{\rho_i; r, s}$ has linear growth, i.e.

$$\|h_i\|_{\rho_\infty; r, s} \leq e^{P_r(i)}$$

for some polynomial P_r . By (8.18) and (8.32), we have

$$\|h_i\|_{\rho_\infty; s+1, s} \leq K(s+2, \theta_i) \tau_i^{\kappa_i}.$$

We now consider the convergence of $\tilde{h}_i^\lambda := \tilde{H}_i^\lambda - \tilde{H}_{i-1}^\lambda = h_i \circ \tilde{H}_{i-1}^\lambda$. We have

$$\begin{aligned} \|\{h_i^i \circ \tilde{H}_{i-1}^\lambda \circ \cdots \circ H_0^\lambda\}\|_{\frac{1}{2}\rho_\infty; r, s} &\leq C_r^{i+1} \left\{ \|h_i\|_{\rho_i; r, s} \right. \\ &\quad + |h_i|_{\rho_i; 1, 0} \sum_{j \leq k < i} \|h_j\|_{\rho_j; s+1, s} \odot \|h_k\|_{\rho_k; r, s_*} + |h_i|_{\rho_i; 1, s_*} \sum_{j < i} \|h_j\|_{\rho_j; r, s} \\ &\quad \left. + \sum_{j < i} \left\{ \|h_i\|_{\rho_i; s+1, s} \odot \|h_j\|_{\rho_j; r, s_*} + \|h_j\|_{\rho_j; s+1, s} \odot \|h_i\|_{\rho_{i+1}; r, s_*} \right\} \right\}. \end{aligned}$$

This shows that $\|\tilde{h}_i\|_{\rho_\infty/2; r, s}$ has linear growth. The above formula also holds when $r = s + 1$. Then we see that $\|\tilde{h}_i\|_{\rho_\infty/2; s+1, s}$ converges rapidly. We now consider intermediate Hölder estimates. Assume that $s + 1 < \ell < r$. By interpolation in x variables, we have

$$|\tilde{h}_i|_{\rho_\infty/2; \ell-i, i} \leq C_r |\tilde{h}_i|_{\rho_\infty/2; s+1-i, i}^\theta |\tilde{h}_i|_{\rho_\infty/2; r-i, i}^{1-\theta}$$

for positive numbers $\theta = \frac{r-\ell}{r-s-1}$. This shows that \tilde{H}_i^λ converges in $\mathcal{C}^{\ell, s}$ norm for any $\ell < r$. That \tilde{H}_i^λ converges to a $\mathcal{C}^{1,0}$ diffeomorphism can be proved in a way similar to the proof of Proposition 1.2.

(ii) Assume that $r = \infty > s$. We first apply the above arguments to $r = s + 3$. This gives us the rapid convergence of $\|\tilde{h}_i\|_{\rho_i; s+1, s}$. We also apply the above estimates to any finite $r = s + \ell$ for any integer $\ell \geq 3$. Then $\|\tilde{h}_i\|_{\rho_\infty/2; r, s}$ has linear growth. By interpolation in x variables as before, we get rapid convergence of $\|\tilde{h}_i\|_{\rho_\infty/2; [s]+\ell-1, s}$ for $\ell = 3, 4, \dots$.

We remark that the case $r = s = \infty$ needs to be treated differently as (8.24) is too weak to conclude that $\|\mathcal{A}_i\|_{\rho_\infty; r, s}$ has linear growth as we cannot have (8.15) for all s . Therefore, we will treat the case in (iv).

(iii) Assume that s is finite. Since

$$d_{[s]}(r, [s] + 1) > 1,$$

we can apply the above arguments in the first case to get rapid convergence of \tilde{h}_i in $\mathcal{C}^{r', [s]}$ norm for any $r' < r$. Since \tilde{h}_i has linear growth in $\mathcal{C}^{r, s}$ norm. By interpolation in t variable

$$|\tilde{h}_i|_{\rho_\infty/2; [r]-i, i+\beta} \leq C_r |\tilde{h}_i|_{\rho_\infty/2; [r]-i, i}^{1-\beta/\{s\}} |\tilde{h}_i|_{\rho_\infty/2; [r]-i, i+\{s\}}^{\beta/\{s\}}.$$

We obtain the rapid convergence of $|\tilde{h}_i|_{\rho_\infty/2; [r]-j, j+\beta}$ for any $\beta < \{s\}$.

(iv) We consider the case with $\infty \geq r = s > 1$. We will apply smoothing operator in all variables, including the parameter λ . This requires us to re-investigate the use of the homotopy formula. Recall that $D_\rho = B_\rho^{2n} \times B_\rho^M \times B_\rho^L$ and $\tilde{\rho}_i = (1 - \theta)^i \rho$. We will assume that $1/4 < \rho < 4$. Set $|u|_{\rho; a} = |u|_{D_\rho; a}$ and $|\mathcal{A}|_{\rho; a} = |\mathcal{A}|_a$.

We first assume that r is finite. We use a smoothing operator S_τ in (z, t, ξ, λ) variables on $D_\rho \times (-1, 2)$. We first apply the Seeley extension operator E in parameter λ and then S_τ to define

$$h_\tau = (f_\tau, g_\tau) := -S_\tau E \mathcal{T}_\rho \mathcal{A} + S_\tau E \mathcal{T}_\rho \mathcal{A}(0).$$

On shrinking domains $D_{\tilde{\rho}_1} \times [0, 1]$, we have

$$[S_\tau, \mathbf{d}] = 0, \quad [E, \mathbf{d}] = 0.$$

The vanishing of last commutator follows from the linearity of the Seeley extension (8.4) which is applied in the λ variable. We obtain for $0 \leq \lambda \leq 1$

$$\begin{aligned}\mathcal{A} + \mathbf{d}h_\tau &= \mathcal{A} - \mathbf{d}S_\tau E\mathcal{T}_\rho \mathcal{A} = \mathcal{A} - S_\tau E\mathbf{d}\mathcal{T}_\rho \mathcal{A} = \mathcal{A} - S_\tau E\mathcal{A} + S_\tau E\mathcal{T}_\rho \mathbf{d}\mathcal{A} \\ &= (I - S_\tau)E\mathcal{A} + S_\tau E\mathcal{T}_\rho \mathbf{d}\mathcal{A}.\end{aligned}$$

Here the last identity is obtained by the homotopy formula and $E^\lambda u^\lambda = u^\lambda$ for $0 \leq \lambda \leq 1$. We now express

$$(8.34) \quad (\mathcal{C}\hat{\mathcal{A}}) \circ H = \mathcal{A} + \mathbf{d}h_\tau + \langle \mathcal{A}, \partial h_\tau \rangle = I_1 + I_2 + I_3$$

where $\mathcal{C}\hat{\mathcal{A}} = (C\hat{\mathcal{B}}, \hat{\mathbf{b}})$ and I_i are now given by

$$\begin{aligned}I_1 &= (I - S_\tau)E\mathcal{A}, \quad I_2 = S_\tau E\mathcal{T}_\rho \mathbf{d}\mathcal{A}, \quad I_3 = \langle \mathcal{A}, \partial S_\tau E\mathcal{T}_\rho \mathcal{A} \rangle, \\ C \circ H - I &= I_4, \quad I_4 = \langle \mathcal{A}, \partial S_\tau E\mathcal{T}_\rho \mathcal{A} \rangle.\end{aligned}$$

Here $I_{4,\bar{\alpha}\beta} = -(A_{\bar{\alpha}\gamma}\partial_\gamma + B_{\bar{\alpha}\ell}\partial_\ell'')S_\tau E(\mathcal{T}_\rho \mathcal{A})_\beta$. Each I_i contains the extension and smoothing operators. By (7.20), the condition has the form

$$\mathbf{d}\mathcal{A}^\lambda = \langle \psi(\mathcal{A}^\lambda)\mathcal{A}^\lambda, \partial \mathcal{A}^\lambda \rangle, \quad 0 \leq \lambda \leq 1.$$

We will use

$$\hat{\mathcal{A}} \circ H = (I + I_4)^{-1}(I_1 + I_2 + I_3).$$

From (8.4), we know that the extension E does not depend on the domain D_ρ . Thus we have $|Eu|_r \leq C_r|u|_{\rho;r}$, where $|Eu|_r$ is computed on $\mathbf{R}^N \times (-1, 2)$ and C_r does not depend on shrinking domains. Although the extension operator does not preserves the integrability condition when λ is outside $[0, 1]$, we still have

$$|EP_\rho \mathbf{d}\mathcal{A}|_{\tilde{\rho}_1;a} \leq C_a|P_\rho \mathbf{d}\mathcal{A}|_{\tilde{\rho}_1;a} \leq C_a(\rho\theta)^{-a}|\langle \psi(\mathcal{A}^\lambda)\mathcal{A}^\lambda, \partial \mathcal{A}^\lambda \rangle|_{\rho;a}.$$

This suffices our proof. The reader is also referred to Nijenhuis and Woolf [27] where integrability condition is relaxed to differential inequalities.

The rest of the proof is much simpler. We can simplify the iteration procedure. In a simpler way we will establish the rapid convergence in the \mathcal{C}^0 norms. This will avoid loss of regularity.

Set $|\mathcal{A}|_r = |\mathcal{A}|_{\rho;r}$. Let us start with

$$|I_1|_{\tilde{\rho}_2;0} \lesssim \tau^r |\mathcal{A}|_r, \quad |I_1|_{\tilde{\rho}_2;r} \lesssim |\mathcal{A}|_r.$$

Applying estimates (5.2) and (5.13) for the homotopy formulae, we get

$$|I_2|_{\tilde{\rho}_3;0} = |S_\tau E\mathcal{T}_\rho \mathbf{d}\mathcal{A}|_{\tilde{\rho}_3;0} \leq K(1)|\mathcal{A}|_1|\mathcal{A}|_0 \leq K(1)|\mathcal{A}|_0^{2-\frac{1}{r}}|\mathcal{A}|_r^{\frac{1}{r}},$$

where the last inequality is obtained by interpolation (A.1). We also have

$$\begin{aligned}|I_2|_{\tilde{\rho}_3;r} &\lesssim \tau^{-1}|E\mathcal{T}_\rho \mathbf{d}\mathcal{A}|_{\tilde{\rho}_2;r-1} \leq K(r)\tau^{-1}(|\mathcal{A}|_1|\mathcal{A}|_{r-1} + |\mathcal{A}|_r|\mathcal{A}|_0) \\ &\leq K(r)\tau^{-1}|\mathcal{A}|_r|\mathcal{A}|_0.\end{aligned}$$

By (8.5) and (8.7), we have

$$\begin{aligned}|DS_\tau E\mathcal{T}_\rho \mathcal{A}|_{\tilde{\rho}_3;0} &\leq |S_\tau E\mathcal{T}_\rho \mathcal{A}|_{\tilde{\rho}_3;1} \leq C\tau^{-1}|E\mathcal{T}_\rho \mathcal{A}|_{\tilde{\rho}_2;0} \leq K(0)\tau^{-1}|\mathcal{A}|_0, \\ |DS_\tau E\mathcal{T}_\rho \mathcal{A}|_{\tilde{\rho}_3;r} &\leq |S_\tau E\mathcal{T}_\rho \mathcal{A}|_{\tilde{\rho}_3;r+1} \leq K(r)\tau^{-1}|\mathcal{A}|_r.\end{aligned}$$

Thus by (A.2), we get for $i = 3, 4$

$$\begin{aligned} |I_i|_{\tilde{\rho}_3;0} &\leq C|\langle \mathcal{A}, DS_\tau E\mathcal{T}_\rho \mathcal{A} \rangle|_{\tilde{\rho}_3;0} \leq K(1)\tau^{-1}|\mathcal{A}|_0^2, \\ |I_i|_{\tilde{\rho}_3;r} &\leq C_r|\langle \mathcal{A}, DS_\tau E\mathcal{T}_\rho \mathcal{A} \rangle|_{\tilde{\rho}_3;r} \leq K(r)\tau^{-1}|\mathcal{A}|_r|\mathcal{A}|_0. \end{aligned}$$

Therefore,

$$\begin{aligned} (8.35) \quad |(C\hat{\mathcal{A}}) \circ H|_{\tilde{\rho}_3;0} &\leq K(r) \left(\tau^r |\mathcal{A}|_r + |\mathcal{A}|_0^{2-\frac{1}{r}} |\mathcal{A}|_r^{\frac{1}{r}} + \tau^{-1} |\mathcal{A}|_0^2 \right), \\ |(C\hat{\mathcal{A}}) \circ H|_{\tilde{\rho}_3;r} &\leq K(r) (|\mathcal{A}|_r + \tau^{-1} |\mathcal{A}|_0 |\mathcal{A}|_r). \end{aligned}$$

Assume that $r > 1$. We first require that $|I_i|_{\tilde{\rho}_3;0} \leq 1/C_N$. Thus we assume that

$$(8.36) \quad \tau^{-1} |\mathcal{A}|_0 + |\mathcal{A}|_0^{2-\frac{1}{r}} |\mathcal{A}|_r^{\frac{1}{r}} \leq 1/(C_N K(1)).$$

By the product rule in Lemma 3.1, (8.35), and above estimates for I_i , we get

$$\begin{aligned} (8.37) \quad |\hat{\mathcal{A}} \circ H|_{\tilde{\rho}_3;0} &\leq K(r) \left(\tau^r |\mathcal{A}|_r + |\mathcal{A}|_0^{2-\frac{1}{r}} |\mathcal{A}|_r^{\frac{1}{r}} + \tau^{-1} |\mathcal{A}|_0^2 \right), \\ |\hat{\mathcal{A}} \circ H|_{\tilde{\rho}_3;r} &\leq K(r) (|\mathcal{A}|_r + \tau^{-1} |\mathcal{A}|_0 |\mathcal{A}|_r) \leq 2K(r) |\mathcal{A}|_r. \end{aligned}$$

Thus $h = H - I$ satisfies

$$(8.38) \quad |h|_{\tilde{\rho}_2;m} = |S_\tau \mathcal{T}_\rho \mathcal{A}|_{\tilde{\rho}_2;m} \leq K(m) |\mathcal{A}|_{\rho;m}.$$

Interpolating again, we have $|\mathcal{A}|_1 \leq C_r |\mathcal{A}|_0^{1-\frac{1}{r}} |\mathcal{A}|_r^{\frac{1}{r}}$. Assume now that

$$(8.39) \quad |\mathcal{A}|_0^{1-\frac{1}{r}} |\mathcal{A}|_r^{\frac{1}{r}} \leq K_0^{-1}(1) \theta / C_N$$

with $K_0(1) \geq K(1)$. Then $|h|_{\tilde{\rho}_2;1} \leq \theta / C_N$, and by (A.20) we obtain

$$|H^{-1} - I|_{\tilde{\rho}_3;1} \leq 2|H - I|_{\tilde{\rho}_2;1} \leq K(1) |\mathcal{A}|_0^{1-\frac{1}{r}} |\mathcal{A}|_r^{\frac{1}{r}} \leq 1/2.$$

By (6.33) and $H^{-1}: D_{\tilde{\rho}_4} \rightarrow D_{\tilde{\rho}_3}$, we have

$$(8.40) \quad |\hat{\mathcal{A}}|_{\tilde{\rho}_4;r} \lesssim |\hat{\mathcal{A}} \circ H|_{\tilde{\rho}_3;r} + |\hat{\mathcal{A}} \circ H|_{\tilde{\rho}_3;1} |h|_{\tilde{\rho}_3;r} \leq K(r) |\mathcal{A}|_r.$$

By (8.37), we have

$$(8.41) \quad |\hat{\mathcal{A}}|_{\tilde{\rho}_4;0} \leq K(r) \left(\tau^r |\mathcal{A}|_r + |\mathcal{A}|_0^{2-\frac{1}{r}} |\mathcal{A}|_r^{\frac{1}{r}} + \tau^{-1} |\mathcal{A}|_0^2 \right).$$

The rest of proof is analogous to the previous cases. We will be brief. Let

$$\rho_i = \frac{1}{2} + \frac{1}{2(i+1)}, \quad \rho_{i+1} = (1 - \theta_i)^4 \rho_i.$$

Thus, we take $1 - \theta_k^* = (1 - \theta_k)^4$ in Lemma 3.4. We return to the sequence H_i which needs to satisfy (8.27). Let S_i^λ be the sequence of the structures and let \mathcal{A}_i be the adapted 1-forms of S_i^λ . Suppose that (8.36) and (8.39) hold for \mathcal{A}_i , i.e.

$$(8.42) \quad K_0(1) \tau_i^{-1} |\mathcal{A}_i|_{\rho_i;0} \leq 1/C_N, \quad |\mathcal{A}_i|_{\rho_i;0}^{1-\frac{1}{r}} |\mathcal{A}_i|_{\rho_i;r}^{\frac{1}{r}} \leq K_0^{-1}(1) \theta_i / C_N.$$

Then we have by (8.40)-(8.41)

$$(8.43) \quad |\mathcal{A}_{i+1}|_{\rho_{i+1};r} \leq K(r) |\mathcal{A}_i|_{\rho_i;r},$$

$$(8.44) \quad |\mathcal{A}_{i+1}|_{\rho_{i+1};0} \leq K(r) \left(\tau_i^r |\mathcal{A}_i|_{\rho_i;r} + |\mathcal{A}_i|_{\rho_i;0}^{2-\frac{1}{r}} |\mathcal{A}_i|_{\rho_i;r}^{\frac{1}{r}} + \tau_i^{-1} |\mathcal{A}_i|_{\rho_i;0}^2 \right).$$

We apply Lemma 8.3 to $a_i := |\mathcal{A}_i|_{\rho_i;0}$ and $L_i := |\mathcal{A}_i|_{\rho_i;r}$. By (8.12) we get

$$(8.45) \quad a_i \leq a_i^* := \tau_i^{\kappa_*}, \quad L_i \leq L_i^* := 2^i L_0 e^{P_1(i)},$$

provided $\tau_0 \leq \hat{\tau}_0$. Here $\kappa_* > 1$ and it depends only on r . Recall from Proposition 8.3 that $\tau_{i+1} = \tau_i^{d_*}$ for some $d_* > 1$. We may choose τ_0 so small that (8.45) implies (8.28) and (8.42). Therefore, using the initial normalization and dilation we first obtain (8.42) for $i = 0$. Then we have (8.43)-(8.44) for $i = 0$. By Lemma 8.3, we have (8.45) for $i = 0, 1$ and hence (8.42) for $i = 0, 1$. We repeat this procedure to get (8.42)-(8.44) for all i . Thus each H_i satisfies (8.38), which now has the form

$$(8.46) \quad |h_i|_{(1-\theta_i)^2 \rho_i; m} \leq K(m) |\mathcal{A}_i|_{\rho_i; m}, \quad m \leq r.$$

By (8.45) and interpolation, we get rapid convergence of $|\mathcal{A}_i|_{\rho_i; m}$ for any $m < r$. Hence we have rapid convergence of $|\tilde{h}_i|_{\rho_\infty/2; m}$ for any $m < r$. This proves the result when r is finite.

When $r = \infty$, we apply the above argument to $r = 2$ to obtain the rapid convergence of $|\tilde{h}_i|_{\rho_\infty/2; a}$ for any $a < 2$. We still have (8.43) and (8.46) for any finite r , which are obtain via (8.40)-(8.41). Thus, \tilde{h}_i has linear growth in \mathcal{C}^r norm for any finite r . By interpolation, we obtain the rapid convergence of \tilde{h}_i in \mathcal{C}^m norm for any $m < r$.

(v) In this case, we will applying smoothing operator to the variables $x \in \mathbf{R}^N$ only. However, the argument is identical to the proof of (iv). \square

APPENDIX A. HÖLDER NORMS FOR FUNCTIONS WITH PARAMETER

The main purpose of the appendix is to derive some interpolation properties for Hölder norms defined in domains depending on a parameter. The interpolation properties were derived in Hörmander [18] and Gong-Webster [10].

We say that a domain D in \mathbf{R}^m has the *cone property* if the following hold: (i) Given two points p_0, p_1 in D there exists a piecewise C^1 curve $\gamma(t)$ in D such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$, $|\gamma'(t)| \leq C_* |p_1 - p_0|$ for all t except finitely many values. The diameter of D is less than C_* . (ii) For each point $x \in \overline{D}$, D contains a cone V with vertex x , opening $\theta > C_*^{-1}$ and height $h > C_*^{-1}$. We will denote $C_*(D)$ a constant $C_* > 1$ satisfying (i) and (ii). A constant $C_a(D)$ may also depend on $C_*(D)$. In our applications, we will apply the inequalities to domains that are products of balls of which the radii are between two fixed numbers. Therefore, $C_*(D)$ and $C_a(D)$ do not depend on D , which will be assumed in the appendix.

If D is a domain of the cone property, then the norms on \overline{D} satisfy

$$(A.1) \quad |u|_{D; (1-\theta)a + \theta b} \leq C_{a,b} |u|_{D; a}^{1-\theta} |u|_{D; b}^\theta,$$

$$|f_1|_{a_1+b_1} |f_2|_{a_2+b_2} \leq C_{a,b} (|f_1|_{a_1+b_1+b_2} |f_2|_{a_2} + |f_1|_{a_1} |f_2|_{a_2+b_1+b_2}).$$

Here $|f_i|_{a_i} = |f_i|_{D_i; a_i}$. Throughout the appendix, we always assume that domains D, D', D_i have the cone property.

Let $|u_i|_{a_i} = |u_i|_{D_i; a_i}$. Then we have

$$(A.2) \quad \prod_{j=1}^m |u_j|_{d_j+a_j} \leq C_a^m \sum_{j=1}^m |u_j|_{d_j+a_1+\dots+a_m} \prod_{i \neq j} |u_i|_{d_i}.$$

Let $f = \{f^\lambda\}$, $g = \{g^\lambda\}$ be two families of functions on \overline{D} and $\overline{D'}$, respectively. To deal with two Hölder exponents in x, t variables, we introduce notation

$$\begin{aligned}
 & |f|_{D;a,0} \odot |g|_{D';0,b} := |f|_{D;a,0} |g|_{D';0,[b]} + |f|_{D;[a],0} |g|_{D';0,b}, \\
 & Q_{D,D';a,b}^*(f, g) := |f|_{D;a,0} \odot |g|_{D';0,b} + |f|_{D;0,b} \odot |g|_{D';a,0}, \\
 (A.3) \quad & Q_{D,D';r,s}(f, g) := \sum_{j=0}^{\lfloor s \rfloor} Q_{D,D';r-j,j+\{s\}}^*(f, g), \quad [r] \geq [s], \\
 & \hat{Q}_{D,D';r,s}(f, g) := \|f\|_{D;r,s} |g|_{D';0,0} + |f|_{D;0,0} \|g\|_{D';r,s} + Q_{D,D';r,s}(f, g).
 \end{aligned}$$

For simplicity, the dependence of Q, Q^*, \hat{Q} on domains D, D' is not indicated when it is clear from the context.

Throughout the paper, by $A \lesssim B$ we mean that $A \leq CB$ for some constant C .

Lemma A.1. *Let r_i, s_i, a_i, b_i ($1 \leq i \leq m$) be non-negative real numbers. Assume that $(r_1, \dots, r_m, a_1, \dots, a_m) \in \mathbf{N}^{2m}$, or $(s_1, \dots, s_m, b_1, \dots, b_m) \in \mathbf{N}^{2m}$. Let D_i be a domain in \mathbf{R}^{n_i} with the cone property and let $f_i = \{f_i^\lambda\}$ be a family of functions on D_i . Let $|f_i|_{c,d} = |f_i|_{D_i;c,d}$. Assume that $\infty > m \geq 2$. Then*

$$(A.4) \quad \prod_{i=1}^m |f_i|_{r_i, s_i} \leq C_{r+s}^m \left\{ \sum_i |f_i|_{r,s} \prod_{\ell \neq i} |f_\ell|_{0,0} + \sum_{i \neq j} Q_{r,s}^*(f_i, f_j) \prod_{\ell \neq i,j} |f_\ell|_{0,0} \right\},$$

$$\begin{aligned}
 (A.5) \quad \prod_{i=1}^m |f_i|_{r_i+a_i, s_i+b_i} & \leq C_{r+s+a+b}^m \left\{ \sum_i |f_i|_{r_i+a, s_i+b} \prod_{\ell \neq i} |f_\ell|_{r_\ell, s_\ell} \right. \\
 & \quad \left. + \sum_{i \neq j} |f_i|_{r_i+a, s_i} |f_j|_{r_j, s_j+b} \prod_{\ell \neq i,j} |f_\ell|_{r_\ell, s_\ell} \right\}.
 \end{aligned}$$

Here $a = \sum a_i$, $b = \sum b_i$, etc.. Assume further that $r_i \geq s_i$ and $a_i = b_i = 0$ for all i . Then

$$(A.6) \quad \|f_1\|_{r_1, s_1} \cdots \|f_m\|_{r_m, s_m} \leq C_{r,s}^m \sum_{i \neq j} \hat{Q}_{r,s}(f_i, f_j) \prod_{\ell \neq i,j} |f_\ell|_{0,0}.$$

Proof. Let $k_i, j_i \in \mathbf{N}$. Let $\partial_{x_i}^{k_i} \partial_{\lambda_i}^{j_i} f_i$ denote a partial derivative of $f^\lambda(x)$ of order k_i in x and order j_i in λ , evaluated at $x = x_i$ and $\lambda = \lambda_i$. Let $k = \sum k_i$ and $j = \sum j_i$.

We will prove the inequality by estimating derivatives pointwise and Hölder ratios of derivatives. We apply (A.2) for domains D_i to obtain

$$(A.7) \quad \prod |\partial_{x_i}^{k_i} \partial_{\lambda_i}^{j_i} f_i| \lesssim \prod |\partial_{\lambda_i}^{j_i} f_i|_{k_i} \lesssim \sum |\partial_{\lambda_i}^{j_i} f_i|_k \prod_{\ell \neq i} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_0.$$

Estimate each term via pointwise derivatives, we can write

$$(A.8) \quad |\partial_{\lambda_i}^{j_i} f_i|_k \prod_{\ell \neq i} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_0 = |\partial_{x_i}^{k'} \partial_{\lambda_i}^{j_i} f_i| \prod_{\ell \neq i} |\partial_{\lambda_\ell}^{j_\ell} f_\ell(x_\ell)|$$

for some partial derivative $\partial^{k'}$ with order $k' \leq k$. While applying (A.2) for the domain I , we see the last term is bounded by the right-hand side of (A.4). Thus, we have verified (A.4) when r_i, s_j are integers.

Let $\beta_i = \{r_i\}$ and $k_i = [r_i]$, or $\beta_i = 0$ and $k_i \leq [r_i]$. We estimate the product of Hölder ratios

$$\prod_{\beta_i > 0} \frac{|\partial_{y_i}^{k_i} \partial_{\lambda_i}^{j_i} f_i - \partial_{x_i}^{k_i} \partial_{\lambda_i}^{j_i} f_i|}{|y_i - x_i|^{\beta_i}} \times \prod_{\beta_i = 0} |\partial_{x_i}^{k_i} \partial_{\lambda_i}^{j_i} f_i| \lesssim \sum |\partial_{\lambda_i}^{j_i} f_i|_r \prod_{\ell \neq i} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_0.$$

We estimate each term in the last sum by considering pointwise derivatives and Hölder ratios. The former can be estimated by computation similar to (A.7)-(A.8). For the Hölder ratio, applying (A.2) for I we can bound

$$\frac{|\partial_{y_i}^k \partial_{\lambda_i}^{j_i} f_i - \partial_{x_i}^k \partial_{\lambda_i}^{j_i} f_i|}{|y_i - x_i|^\beta} \times \prod_{\ell \neq i} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_0$$

by the right-hand side of (A.4).

To verify (A.5) it suffices to consider the case with $(s_1, \dots, s_m, b_1, \dots, b_m) \in \mathbf{N}^{2m}$. Furthermore, it is easy to reduce it to the case with $s_i = 0$, which we now assume. Let $k_i \leq r_i + a_i$ and $j_i \leq b_i$. We apply (A.2) to domains D_i to obtain

$$(A.9) \quad \prod |\partial_{x_i}^{k_i} \partial_{\lambda_i}^{j_i} f_i| \lesssim \prod_{\ell | k_\ell \leq r_\ell} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_{k_\ell} \sum_{i | k_i > r_i} |\partial_{\lambda_i}^{j_i} f_i|_{k_i + a'} \prod_{\ell | \ell \neq i, k_\ell > r_\ell} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_{a_\ell}$$

with $a' = \sum_{\ell | k_\ell > a_\ell} (k_\ell - r_\ell) \leq a$. Again we estimate each term via pointwise derivative. We can write

$$\left\{ \prod_{\ell | k_\ell \leq r_\ell} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_{k_\ell} \right\} |\partial_{\lambda_i}^{j_i} f_i|_{[k_i + a']} \prod_{\ell | \ell \neq i, k_\ell > a_\ell} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_{[a_\ell]} = |\partial_{x_i}^{k'} \partial_{\lambda_i}^{j_i} f_i| \prod_{\ell \neq i} |\partial_{x_\ell}^{k_\ell} \partial_{\lambda_\ell}^{j_\ell} f_\ell|.$$

While applying (A.2) for the domain I , we see the last term is bounded by the right-hand side of (A.5). For Hölder ratios on the right-hand side of (A.9), let $\beta_i = \{a_i + r'\}$ and $k_i = [a_i + r']$, or $\beta_i = 0$ and $k_i \leq [a_i + r']$; and for $\ell \neq i$, let $\beta_\ell = \{a_\ell\}$ or $\beta_\ell = 0$ and let $k_\ell \leq [a_\ell]$. We estimate the product of Hölder ratios

$$\prod_{i | \beta_i > 0} \frac{|\partial_{y_i}^{k_i} \partial_{\lambda_i}^{j_i} f_i - \partial_{x_i}^{k_i} \partial_{\lambda_i}^{j_i} f_i|}{|y_i - x_i|^{\beta_i}} \times \prod_{i | \beta_i = 0} |\partial_{x_i}^{k_i} \partial_{\lambda_i}^{j_i} f_i| \lesssim \sum |\partial_{\lambda_i}^{j_i} f_i|_r \prod_{\ell \neq i} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_0.$$

We need to estimate each term in the last sum by considering pointwise derivatives and Hölder ratios. Applying (A.2) to D_i , we can bound $|\partial_{\lambda_i}^{j_i} f_i|_{[r]} \prod_{\ell \neq i} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_0$ by the right-hand side of (A.5). Let $\beta = \{r\}$ and $k = [r]$. Applying (A.2) to the interval I , we can bound

$$\frac{|\partial_{y_i}^k \partial_{\lambda_i}^{j_i} f_i - \partial_{x_i}^k \partial_{\lambda_i}^{j_i} f_i|}{|y_i - x_i|^\beta} \times \prod_{\ell \neq i} |\partial_{\lambda_\ell}^{j_\ell} f_\ell|_0$$

by the right-hand side of (A.5). We have verified (A.5) and hence (A.6). \square

By the product rule and Lemma A.1, we obtain the following.

Lemma A.2. Let D be a domain in \mathbf{R}^n with the cone property. Assume that $1 \leq m' < m < \infty$. With all norms on D , we have

$$\begin{aligned} & |\{f_1^\lambda \cdots f_m^\lambda\}|_{r,s} + Q_{r,s}^*(\{\prod_{i \leq m'} f_i^\lambda\}, \{\prod_{i > m'} f_i^\lambda\}) \\ & \leq C_{r,s}^m \left\{ \sum_i |f_i|_{r,s} \prod_{k \neq i} |f_k|_{0,0} + \sum_{i \neq j} Q_{r,s}^*(f_i, f_j) \prod_{k \neq i,j} |f_k|_{D;0,0} \right\}. \end{aligned}$$

Let us use Lemmas A.1 and A.2 to verify the following.

Proposition A.3. Let $r, s \in [0, \infty)$. Let ϕ_i be \mathcal{C}^{r+s+1} functions in $[a_i, b_i]$. Let D be as in Lemma A.2. Let $f_i \in C_*^{r,s}(\overline{D})$ and $f_i^\lambda(D) \subset [a_i, b_i]$. Suppose that $0 \in [a_i, b_i]$ and $\phi_i(0) = 0$. Let $1 \leq m' < m$. With all norms on D , we have

$$(A.10) \quad |\phi_1(f_1)|_{r,s} \leq C_{r,s}(|f_1|_{r,s} + |f_1|_{r,0} \odot |f_1|_{0,s}),$$

$$(A.11) \quad |\phi_1(f_1) \cdots \phi_m(f_m)|_{r,s} + Q_{r,s}^*(\prod_{i \leq m'} \phi_i(f_i), \prod_{i > m'} \phi_i(f_i)) \\ \leq C_{r,s}^m \left\{ \sum_i |f_i|_{r,s} \prod_{k \neq i} |f_k|_{0,0} + \sum_{i \leq j} Q_{r,s}^*(f_i, f_j) \prod_{k \neq i,j} |f_k|_{0,0} \right\}.$$

where $C_{r,s}$ also depends on $|\phi_i|_{[a_i, b_i]; [r]+[s]+1}$.

Proof. Since $\phi_i \in C^1$ and $\phi_i(0) = 0$, we get $|\phi_i(f_i)|_{D; \alpha, \beta} \lesssim |f_i|_{\alpha, \beta}$ for $\alpha, \beta \in [0, 1]$. Applying the chain rule and Lemma A.1, we get

$$|\phi_i(f_i)|_{a,b} \lesssim \sum_{a_1 + \cdots + a_m = a, b_1 + \cdots + b_m = b} \prod |f_i|_{a_\ell, b_\ell} \lesssim |f_i|_{a,b} + |f_i|_{a,0} \odot |f_i|_{0,b}$$

with all a_i, b_j are integers, except for possible one. We can verify (A.11) similarly by using Lemma A.1. \square

When applying the chain rule, we need to count derivatives efficiently.

Definition A.4. Let $k \geq 1$ and $k \geq j \geq 0$. Let F^λ be a family of mappings from D to D' . Let $\{u^\lambda\}$ be a family of functions on D and let $\{f_i^\lambda\}, \dots, \{f_m^\lambda\}$ be families of functions on D' . Define $\mathcal{P}_{k,j}(\{u^\lambda \circ F^\lambda\}; \{f^\lambda\})$ to be the linear space spanned by functions of the form

$$(A.12) \quad \{\partial^{K_0} \partial_\lambda^{j_0} u^\lambda\} \circ F^\lambda, \quad \{\partial^{K_0} \partial_\lambda^{j_0} u^\lambda\} \circ F^\lambda \prod_{1 \leq \ell \leq l} \partial^{K_\ell} \partial_\lambda^{j_\ell} f_{n_\ell}^\lambda, \quad 1 \leq l < \infty$$

with

$$(A.13) \quad |K_\ell| + j_\ell \geq 1, \quad \ell \geq 0; \quad \sum_{\ell \geq 0} j_\ell \leq j,$$

$$(A.14) \quad |K_0| + j_0 + \sum_{\ell \geq 1} (|K_\ell| + j_\ell - 1) \leq k, \quad i.e. \quad \sum_{\ell \geq 0} (|K_\ell| + j_\ell - 1) < k.$$

Analogously, define $\mathcal{P}_{k,j}(\{f^\lambda\})$ to be the linear space spanned by

$$(A.15) \quad \prod_{1 \leq \ell \leq l} \partial^{K_\ell} \partial_\lambda^{j_\ell} f_{n_\ell}^\lambda, \quad 1 \leq l < \infty$$

with $|K_\ell| + j_\ell \geq 1, \sum j_\ell \leq j, \sum (|K_\ell| + j_\ell - 1) < k$.

By counting efficiently, we count one less for the order of derivative $\partial^{K_\ell} \partial_\lambda^{j_\ell} f_{n_\ell}^\lambda$. Also the l in (A.12) and (A.15) will have an upper bound depending on k, j .

It is easy to see that if $\{a^\lambda\} \in \mathcal{P}_{k_0, j_0}(\{f^\lambda\})$ and $\{v^\lambda\} \in \mathcal{P}_{k_1, j_1}(\{u^\lambda \circ F^\lambda\}; \{f^\lambda\})$, with $k_0 \geq 1$, $k_1 \geq 1$, then $\{a^\lambda v^\lambda\} \in \mathcal{P}_{k_0+k_1-1, j_0+j_1}(\{u^\lambda \circ F^\lambda\}; \{f^\lambda\})$ by (A.14). We express it as

$$(A.16) \quad \mathcal{P}_{k_0, j_0}(\{f^\lambda\}) \times \mathcal{P}_{k_1, j_1}(\{u^\lambda \circ F^\lambda\}; \{f^\lambda\}) \subset \mathcal{P}_{k_0+k_1-1, j_0+j_1}(\{u^\lambda \circ F^\lambda\}; \{f^\lambda\})$$

for $k_0, k_1 \geq 1$. Also, if $F^\lambda = I + f^\lambda$, then

$$\mathcal{P}_{k, j}(\{u^\lambda \circ F^\lambda\}; \{f^\lambda\}) = \mathcal{P}_{k, j}(\{u^\lambda \circ F^\lambda\}; \{f^\lambda\}).$$

Lemma A.5. *If $1 \leq |K| + j \leq k$ then $\partial^K \partial_\lambda^j \{u^\lambda \circ F^\lambda\} \in \mathcal{P}_{k, j}(\{u^\lambda \circ F^\lambda\}; \{f^\lambda\})$.*

Proof. Let $y = F^\lambda(x)$. Let ∂^i be a derivative of order i in x . We use the chain rule. Applying ∂ or ∂_λ one by one, we can verify that $\partial^{k-j} \partial_\lambda^j (u^\lambda \circ G^\lambda)$ is a linear combination of functions

$$(A.17) \quad (\partial_\lambda^{j_0} \partial^{K_0} u^\lambda) \circ G^\lambda(x) \prod_{1 \leq i \leq m', j_i > 0} \partial_\lambda^{j_i} \partial^{K_i} G_{n_i}^\lambda \prod_{m' < i \leq |K_0|, |K_i| > 0} \partial^{K_i} G_{n_i}^\lambda.$$

Here $\sum_{\ell \geq 0} j_\ell = j$, and $j + \sum_{\ell \geq 1} |K_\ell| = k$. When $|K_0| = 0$, it is clear that (A.13)-(A.14) hold. When $|K_0| > 0$, we have $\sum_{\ell \geq 0} (j_\ell + |K_\ell| - 1) = k - |K_0| < k$. Thus (A.17) is in $\mathcal{P}_{k, j}(\{u^\lambda \circ F^\lambda\}; \{f^\lambda\})$. \square

Define

$$\begin{aligned} \tilde{Q}_{r, s}(\partial f, \partial g) &:= Q_{r-1, s}(\partial f, \partial g), \\ \tilde{Q}_{r, s}(\partial f, \tilde{\partial} g) &:= \tilde{Q}_{r, s}(\partial f, \partial g) + Q_{r-1, s-1}(\partial f, \partial_\lambda g) := \tilde{Q}_{r, s}(\tilde{\partial} g, \partial f), \\ \tilde{Q}_{r, s}(\tilde{\partial} f, \tilde{\partial} g) &= \tilde{Q}_{r, s}(\partial f, \tilde{\partial} g) + Q_{r-1, s-1}(\partial_\lambda f, \partial g) + Q_{r-1, s-2}(\partial_\lambda f, \partial_\lambda g), \end{aligned}$$

where $Q_{r-1, s} = 0$ for $r < s + 1$, $Q_{r-1, s-1} = 0$ for $s < 1$, and $Q_{r-1, s-2} = 0$ for $s < 2$.

Lemma A.6. *Let r, s satisfy (2.3). Let D, D' be bounded domains in $\mathbf{R}^m, \mathbf{R}^n$ respectively, which have the cone property. Let $G^\lambda: D \rightarrow D'$ be of class $\mathcal{C}^{r, s}(\bar{D})$.*

(i) *Assume that $|G|_{D; 1, 0} < 2$. Then*

$$(A.18) \quad |\{u^\lambda \circ G^\lambda\}|_{D; \alpha, 0} \leq C |u|_{D'; \alpha, 0}, \quad 0 \leq \alpha < 1,$$

$$(A.19) \quad |\{u^\lambda \circ G^\lambda\}|_{D; r, 0} \leq C_r \left(|u|_{D'; r, 0} + |\partial u|_{D'; 0, 0} |\partial G|_{D; r-1, 0} \right), \quad r \geq 1.$$

(ii) *Assume that $|G|_{D; 1, 0} \leq 2$ and $s \geq 1$. Then*

$$\begin{aligned} \|\{u^\lambda \circ G^\lambda\}\|_{D; r, s} &\leq C_r \left\{ \|u\|_{D'; r, s} + \tilde{Q}_{D', D; r, s}(\partial u, \tilde{\partial} G) \right. \\ &\quad \left. + |u|_{D'; 1, 0} (|G|_{D; r, s} + \tilde{Q}_{D', D'; r, s}(\tilde{\partial} G, \tilde{\partial} G)) \right\}. \end{aligned}$$

Proof. (i). Inequality (A.18) is immediate and (A.19) is in [18] and [10].

(ii). Let $\alpha = \{r\}$ and $\beta = \{s\}$. Let $r = k + \alpha$ with $k \geq 1$. Let $y = G^\lambda(x)$. Let $\partial^i u^\lambda(y)$ be a partial derivatives in y of order i . We know that $\partial^{k-j} \partial_\lambda^j (u^\lambda \circ G^\lambda) \in \mathcal{P}_{k, j}(\{u^\lambda \circ G^\lambda\}; \{G^\lambda\})$, i.e. it is a linear combination of functions $v^\lambda(x)$ of the form (A.17). We can express v^λ as

$$v^\lambda(x) = (\partial_\lambda^{j_0} \partial^m u^\lambda) \circ G^\lambda(x) \prod_{1 \leq i \leq m'} \partial_\lambda^{j_i} \partial^{k_i} \partial_\lambda G_{n_i}^\lambda \prod_{m' < i \leq m} \partial^{k_i} \partial G_{n_i}^\lambda,$$

where $j = j_0 + \dots + j_{m'} + m'$, and by (A.14) we have $j + m + \sum k_i \leq k$.

When $m = 0$, it is immediate that by $|G|_1 < 2$, we obtain $|v|_{\alpha,\beta} \lesssim |u|_{\alpha,j+\beta}$. Suppose that $m \geq 1$. Computing the Hölder norms, we get

$$\begin{aligned} |v|_{\alpha,\beta} &\lesssim |\partial u|_{m-1+\alpha,j_0+\beta} \prod |\partial_\lambda G_{n_i}|_{k_i,j_i} \prod |\partial G_{n_i}|_{k_i,0} \\ &\quad + \sum_\ell |\partial u|_{j_0+m-1,j_0} |\partial_\lambda G_{n_\ell}|_{m_\ell+\alpha,j_\ell+\beta} \prod_{i \neq \ell, i, \ell \leq m'} |\partial_\lambda G_{n_i}|_{k_i,j_i} \prod_{i > m'} |\partial G_{n_i}|_{k_i,0} \\ &\quad + \sum_\ell |\partial u|_{j_0+m-1,j_0} |\partial G_{n_\ell}|_{m_\ell+\alpha,j_\ell+\beta} \prod_{i \leq m'} |\partial_\lambda G_{n_i}|_{k_i,j_i} \prod_{i \neq \ell, i, \ell > m'} |\partial G_{n_i}|_{k_i,0}. \end{aligned}$$

We use Lemma A.1 and obtain

$$\begin{aligned} |v|_{\alpha,\beta} &\lesssim |\partial u|_{r-1,s} + |u|_{1,0} |G|_{r,s} + Q_{r-1,s}(\partial u, \partial G) + Q_{r-1,s-1}(\partial u, \partial_\lambda G) \\ &\quad + |u|_{1,0} \{Q_{r-1,s}(\partial G, \partial G) + Q_{r-1,s-1}(\partial G, \partial_\lambda G) + Q_{r-1,s-2}(\partial_\lambda G, \partial_\lambda G)\}. \quad \square \end{aligned}$$

Let $D_\rho = B_\rho^{2n} \times B_\rho^M \times B_\rho^L$ with $N = 2n + K + L$, and set $|\cdot|_{\rho;r,s} = |\cdot|_{D_\rho;r,s}$ and $\tilde{Q}_{\rho,\tilde{\rho};r,s} = \tilde{Q}_{D_\rho,D_{\tilde{\rho}};r,s}$.

Lemma A.7. *Let $1 \leq r < \infty$. Let $0 < \rho < \infty$, $0 < \theta < 1/2$ and $\rho_i = (1 - \theta)^i \rho$. Let $F^\lambda = I + f^\lambda$ be mappings from D_ρ into \mathbf{R}^N . Assume that $F \in \mathcal{C}^{1,0}(\overline{D_\rho})$ and*

$$f^\lambda(0) = 0, \quad |f|_{\rho;1,0} \leq \theta/C_N.$$

(i) *Then F^λ are injective in D_ρ . There exist unique $G^\lambda = I + g^\lambda$ satisfying*

$$G^\lambda: D_{\rho_1} \rightarrow D_\rho, \quad F^\lambda \circ G^\lambda = I \quad \text{on } D_{\rho_1}.$$

Furthermore, $F^\lambda: D_\rho \rightarrow D_{(1-\theta)^{-1}\rho}$ and $G^\lambda \circ F^\lambda = I$ in D_{ρ_2} .

(ii) *Suppose that $1/4 < \rho < 2$. Assume further that $F \in \mathcal{C}^{r,0}(D_\rho)$. Then $\{g^\lambda\} \in \mathcal{C}^{r,0}(D_{\rho_1})$ and*

$$\begin{aligned} (A.20) \quad &|g|_{\rho_1;r,0} \leq C_r |f|_{\rho;r,0}, \\ &|u \circ G|_{\rho_1;r,0} \leq C_r (|u|_{\rho;r,0} + |u|_{\rho;1,0} |f|_{\rho;r,0}). \end{aligned}$$

(iii) *Suppose that $1/4 < \rho < 2$. Assume further that r, s satisfy (2.3) with $s \geq 1$, $|f|_{\rho;1} < C$, and $f \in \mathcal{C}^{r,s}(D_\rho)$. Then for $u \in \mathcal{C}^{r,s}(D_\rho)$*

$$(A.21) \quad \|g\|_{\rho_1;r,s} \leq C_r (\|f\|_{\rho;r,s} + \tilde{Q}_{\rho,\rho;r,s}(\partial f, \tilde{\partial} f) + |f|_{\rho;1,0} \tilde{Q}_{\rho,\rho;r,s}(\tilde{\partial} f, \tilde{\partial} f)),$$

$$(A.22) \quad \|u^\lambda \circ G^\lambda\|_{\rho_1;r,s} \leq C_r (\|u\|_{\rho;r,s} + \tilde{Q}_{\rho,\rho_1;r,s}(\partial u, \tilde{\partial} f) + |u|_{\rho;1,0} \tilde{Q}_{\rho,\rho;r,s}(\tilde{\partial} f, \tilde{\partial} f)).$$

Proof. We may reduce the proof to $\rho = 1$ by using dilations $x \rightarrow \rho^{-1} F^\lambda(\rho x)$ and $x \rightarrow \rho^{-1} G^\lambda(\rho x)$. The proof of (i) can be obtained easily by applying the contraction mapping theorem to

$$g^\lambda(x) = -f^\lambda(x + g^\lambda(x)).$$

(ii) is a special case of (iii). Thus, let us verify (iii). Assume that $r > 1$. Differentiating the above identity, we separate terms of the highest order derivatives of g^λ from the rest to get identities

$$(\partial^K \partial_\lambda^j) g_l^\lambda + \sum_m (\partial_{y^m} f_l^\lambda) \circ G^\lambda (\partial^K \partial_\lambda^j) g_m^\lambda = E_{lKj}^\lambda(x).$$

Here E_{lKj}^λ are the linear combinations of the functions in $\mathcal{P}_{|K|+j,j}(\{f^\lambda \circ G^\lambda\}, \{g^\lambda\})$ of the form

$$(A.23) \quad (\partial^{K_0} \partial_\lambda^{j_0} f_l^\lambda, \quad v^\lambda(x) := (\partial^{M_0} \partial_\lambda^{j_0} f_l^\lambda) \circ G^\lambda \prod_{1 \leq \ell \leq m} \partial^{M_\ell} \partial_\lambda^{j_\ell} g_{n_\ell}^\lambda$$

with $m \leq |M_0|$, $|M_\ell| + j_\ell \geq 1$, and $\sum j_\ell \leq j$. Furthermore,

$$(A.24) \quad |M_\ell| + j_\ell < |K| + j, \quad \ell > 0; \quad \sum_{\ell \geq 0} (|M_\ell| + j_\ell - 1) < |K| + j.$$

We now verify that $\partial^K \partial_\lambda^j g^\lambda$ is a finite sum of

$$(A.25) \quad h_{Kj}^\lambda(x) := \{A(\partial f) \partial^{K_1} \partial_\lambda^{j_1} f_{n_1} \cdots \partial^{K_m} \partial_\lambda^{j_m} f_{n_m}\} \circ G^\lambda,$$

where $A(\partial^1 f)$ is a polynomial in $(\det(\partial f^\lambda))^{-1}$ and ∂f^λ . Moreover, $j_i \geq 1$ or $|K_i| + j_i \geq 2$, and

$$(A.26) \quad \sum_{\ell \geq 0} (|K_\ell| + j_\ell - 1) < |K| + j, \quad \sum j_\ell \leq j.$$

The assertion is trivial when $|K| + j = 1$. Assume that it holds for $|K| + j < N$. By (A.16), we can see that the $\{v^\lambda\}$ in (A.23) has the form (A.25). This shows that v^λ is of the form (A.25) and (A.26). The claim has been verified.

The estimation of $|\{h_{Kj}^\lambda\}|_{\{r\},\{s\}}$ is the same as in the proof of Lemma A.6. Indeed, when $|M_0| = 0$, $|\{h_{Kj}^\lambda\}|_{\{r\},\{s\}} \prec \|f\|_{r,s}$ for $|K| + j \leq [r]$ and $j \leq [s]$. When $|M_0| > 0$,

$$|\{h_{Kj}^\lambda\}|_{\{r\},\{s\}} \prec \tilde{Q}_{r,s}(\partial f, \tilde{\partial} f) + |f|_{1,0} \tilde{Q}_{r,s}(\tilde{\partial} f, \tilde{\partial} f).$$

This verifies (A.21).

To verify (A.22), as in (A.23)-(A.24) we note that $\partial^k \partial_\lambda^j (u^\lambda \circ G^\lambda(x))$ is a linear combination of functions in $\mathcal{P}_{k+j,j}(\{u^\lambda \circ G^\lambda\}; \{G^\lambda\})$ of the form

$$(A.27) \quad \tilde{v}^\lambda = (\partial^{M_0} \partial_\lambda^{j_0} u^\lambda) \circ G^\lambda \prod_{1 \leq \ell \leq m} \partial^{M_\ell} \partial_\lambda^{j_\ell} g_{n_\ell}^\lambda$$

with $m \leq |M_0|$, $|M_\ell| + j_\ell \geq 1$, $\sum j_\ell \leq j$, and

$$(A.28) \quad |M_\ell| + j_\ell < k + j, \quad \ell > 0; \quad \sum_{\ell \geq 0} (|M_\ell| + j_\ell - 1) \leq k + j - 1.$$

Expressing $\partial^K \partial_\lambda^j g^\lambda$ via linear combinations in (A.25) satisfying (A.26) and applying them to $\partial^{M_\ell} \partial_\lambda^{j_\ell} g^\lambda$ in (A.27), we conclude that \tilde{v}^λ and hence $\partial^K \partial_\lambda^j g^\lambda$ is a linear combination of

$$\tilde{h}_{Kj}^\lambda(x) := \left\{ A(\partial f) \partial^{M_0} \partial_\lambda^{j_0} u^\lambda \prod_{1 \leq \ell \leq |M_0|} \prod_{i \leq n_\ell} \partial^{M_\ell} \partial_\lambda^{j_\ell} f_{n_\ell^i} \right\} \circ G^\lambda$$

where M_ℓ, j_ℓ still satisfy (A.28). When $|M_0| = 0$, we actually have $\tilde{h}_{Kj}^\lambda(x) = (\partial_\lambda^j u^\lambda) \circ G^\lambda$. It is straightforward that $|\tilde{h}_{Kj}|_{\alpha,\beta} \lesssim |u|_{r,s}$. When $|M_0| > 0$, the estimate for $|\tilde{h}_{Kj}|_{\alpha,\beta}$ is obtained analogous to that of h_{Kj} . \square

We will also need to deal with sequences of compositions.

Proposition A.8. *Let D_m be a sequence of domains in \mathbf{R}^d satisfying the cone property of which the constants $C_*(D_m) > 1$ are bounded. Assume that $D_m \subset D_\infty$ and D_∞ also has the cone property. Let $F_m^\lambda = I + f_m^\lambda: D_m \rightarrow D_{m+1}$ for $m = 1, \dots$. Suppose that*

$$f_i^\lambda(0) = 0, \quad |f_i|_{D_i;1,0} \leq 1.$$

Assume that $r \geq s \geq 1$ and $\{r\} \geq \{s\}$. Then

$$\begin{aligned} \|\{u^\lambda \circ F_\ell^\lambda \circ \dots \circ F_1^\lambda\}\|_{D_1;r,s} &\leq C_r^\ell \left\{ \|u\|_{D_\infty;r,s} + \sum \tilde{Q}_{r,s}(\partial u, \tilde{\partial} f_m) \right. \\ &\quad \left. + |u|_{D_\infty;1,0} \left(\sum \|f_m\|_{D_m;r,s} + \sum_{i \leq j} \tilde{Q}_{r,s}(\tilde{\partial} f_i, \tilde{\partial} f_j) \right) \right\}. \end{aligned}$$

Proof. We first derive the factor C_r^ℓ in the inequality. Let $G_i^\lambda(x) = F_i^\lambda \circ \dots \circ F_1^\lambda$. Let $h_{\ell,k}^\lambda$ be a k -th order derivative of $u^\lambda \circ G_\ell^\lambda$. We express $h_{\ell,k}^\lambda$ as a sum of terms of the form

$$(A.29) \quad h_2^\lambda(x) := \tilde{h}_2^\lambda(x)(\partial_\lambda^{j_0} \partial^{k_0} u^\lambda) \circ G_\ell^\lambda(x) \prod_{1 \leq i \leq T'} \partial_\lambda^{j_i} \partial^{k_i} \partial_\lambda F_{n_i}^\lambda \prod_{T' < i \leq T} \partial^{k_i} \partial F_{n_i}^\lambda.$$

Here $\sum k_i + \sum j_i \leq k - 1$, $T' + \sum j_i \leq j$, and \tilde{h}_2^λ is 1 or a product of first-order derivatives of F_i^λ in x , which may be repeated. Let $T_{\ell,k}$ be the maximum number of derivative functions of $u^\lambda, F_\ell^\lambda, \dots, F_1^\lambda$ that appear in (A.29). We have $T_{\ell,0} = 0$ and $T_{\ell,1} = \ell + 1$. By the chain rule, $T_{\ell,k} \leq T_{\ell,k-1} + \ell \leq k\ell + 1$. Let $N_{\ell,k}$ be the maximum number of terms (A.29) that are needed to express $h_{\ell,k}^\lambda$ as a sum of the monomials (A.29). Note that $u^\lambda(x), F_i^\lambda(x)$ are functions in $d + 1$ variables. By the chain rule $N_{\ell,1} \leq (d + 1)^\ell$. By the product and chain rules,

$$N_{\ell,k} \leq N_{\ell,k-1} T_{\ell,k-1} (d + 1)^\ell \leq (k\ell + 1)(d + 1)^\ell N_{\ell,k-1} < C_k^\ell.$$

For a fixed i , the first-order derivatives of F_i^λ in x cannot be repeated more than k times in h_2^λ . Thus, we have

$$|\tilde{h}_2^\lambda(x)| \leq (1 + |f_1|_{1,0})^k \cdots (1 + |f_\ell|_{1,0})^k < 2^{\ell k}.$$

Also the T in (A.29) is less than k . By Lemma A.2, we have

$$|h_2|_{\alpha,\beta} \leq N_{k+1} 2^{\ell k} \left\{ \|u\|_{r,s} + \sum Q_{r,s}(u, f_i) + |u|_{1,0} \left(\sum |f_i|_{r,s} + \sum Q_{r,s}(f_i, f_j) \right) \right\}.$$

Here N_{k+1} arises from the number of terms when computing the Hölder norms after k derivatives. \square

Next, we consolidate the expressions such as $\tilde{Q}_{r,s}(\partial f, \tilde{\partial} g)$ in Lemma A.7 and Proposition A.8 by a simpler expression $Q_{r,s}(f, g)$, defined by (A.3), and the new expression

$$Q'_{r,s}(f, g) := Q_{r,s}(f, g) + \sum_{j=1}^{\lfloor s \rfloor} Q_{r-j+1, j+\{s\}}^*(f, g), \quad s \geq 1.$$

Lemma A.9. *Let $f \in C^{r,s}(D)$, $g \in C^{r,s}(D')$ with D, D' having the cone property. Then*

$$(A.30) \quad Q_{r-1,s}(\partial f, g) \leq C_r \hat{Q}_{r,s}(f, g), \quad Q_{r-1,s-1}(\partial_\lambda f, g) \leq C_r \hat{Q}_{r,s}(f, g),$$

$$(A.31) \quad Q_{r-1,s}(\partial f, \partial g) \leq C_r \{ |f|_{1,\{s\}} \odot \|g\|_{r,0} + |f|_{1,0} \|g\|_{r,s} \\ + |g|_{1,\{s\}} \odot |f|_{r,0} + |g|_{1,0} |f|_{r,s} + Q'_{r,s}(f, g) \},$$

$$(A.32) \quad Q_{r-1,s-1}(\partial f, \partial_\lambda g) \leq C_r \{ |f|_{1,\{s\}} \odot |g|_{r-1,1} + Q'_{r,s}(f, g) \}, \quad 1 \leq s < 2 \leq r,$$

$$(A.33) \quad Q_{r-1,s-1}(\partial f, \partial_\lambda g) \leq C_r \{ |f|_{1,\{s\}} \odot |g|_{r-1,1} + |f|_{0,1} \|g\|_{r,s-1} \\ + |g|_{0,1} \|f\|_{r,s-1} + Q'_{r,s}(f, g) \}, \quad s \geq 1,$$

$$(A.34) \quad Q_{r-1,s-2}(\partial_\lambda f, \partial_\lambda g) \leq C_r \{ |f|_{0,1} \|g\|_{r,s-1} + |g|_{0,1} \|f\|_{r,s-1} \\ + |f|_{\{r\},s} |g|_{0,0} + |g|_{\{r\},s} |f|_{0,0} + Q'_{r,s}(f, g) \}.$$

Here the norms of f (resp. g) and its derivatives are in D (resp. D').

Proof. We verify the first inequality in (A.30) by $|\partial f|_{r-1-j,0} \odot |g|_{0,j+\{s\}} \leq Q_{r,s}(f, g)$ and

$$|\partial f|_{0,j+\{s\}} \odot |g|_{r-1-j,0} \lesssim |f|_{0,j+\{s\}} \odot |g|_{r-j,0} + |f|_{r-j,j+\{s\}} |g|_{0,0} \lesssim \hat{Q}_{r,s}(f, g).$$

Here the first inequality follows from (A.5). The second inequality in (A.30) follows from $|g|_{r-1-j,0} |\partial_\lambda f|_{0,j+\{s\}} \leq Q_{r,s}(f, g)$ and

$$|g|_{0,j+\{s\}} |\partial_\lambda f|_{r-1-j,0} \lesssim |g|_{0,j+1+\{s\}} |f|_{r-1-j,0} + |g|_{0,0} |f|_{r-1-j,j+1+\{s\}} \lesssim \hat{Q}_{r,s}(f, g).$$

We verify (A.31), by $|\partial f|_{r-1,0} \odot |\partial g|_{0,\{s\}} \leq |f|_{r,0} \odot |g|_{1,\{s\}}$ and for $j > 0$

$$|\partial f|_{r-1-j,0} \odot |\partial g|_{0,j+\{s\}} \leq |\partial f|_{r-j,0} \odot |g|_{0,j+\{s\}} + |\partial f|_{0,0} |g|_{r-j,j+\{s\}}.$$

Inequality (A.32) follows from $|\partial f|_{r-j-1,0} \odot |\partial_t g|_{0,j+\{s\}} \leq Q'_{r,s}(f, g)$ and $|\partial f|_{0,\{s\}} \odot |\partial_t g|_{r-1,0} \leq |f|_{1,\{s\}} \odot |g|_{r-1,1}$. For (A.33), we need to verify extra terms. Suppose $0 < j < [s]$. Then

$$|\partial f|_{0,j+\{s\}} \odot |\partial_t g|_{r-j-1,0} \lesssim |f|_{0,j+\{s\}} \odot |\partial_t g|_{r-j,0} + |f|_{r-j,j+\{s\}} |g|_{0,1}$$

and $|f|_{0,j+\{s\}} \odot |\partial_t g|_{r-j,0} \lesssim |f|_{0,j+1+\{s\}} \odot |g|_{r-j,0} + |f|_{0,1} |g|_{r-j,j+\{s\}}.$

We verify (A.34) by $|\partial_t f|_{\{r\},0} \odot |\partial_t g|_{0,j+\{s\}} \lesssim |f|_{\{r\},0} \odot |g|_{0,s} + |f|_{\{r\},s} |g|_{0,0}$ and

$$|\partial \partial_t f|_{r-j-2,0} \odot |\partial_t g|_{0,j+\{s\}} \lesssim |\partial f|_{r-j-2,0} \odot |g|_{0,j+2+\{s\}} + \|f\|_{r,s-1} |g|_{0,1}. \quad \square$$

We now state simple version of the above inequalities, which suffices applications in this paper. The next two propositions are immediate consequences of Lemma A.7, Proposition A.8, Lemma A.9, and the following crude estimate

$$\tilde{Q}_{r,s}(\tilde{\partial} f, \tilde{\partial} g) \leq C(|f|_{1,s_*} \|g\|_{r,s} + |g|_{1,s_*} \|f\|_{r,s} \\ + \|f\|_{s+1,s} \odot \|g\|_{r,s_*} + \|g\|_{s+1,s} \odot \|f\|_{r,s_*}).$$

Proposition A.10. *Let $s_* = 0$ for $s = 0$ and $s_* = 1$ for $s \geq 1$. Let $F^\lambda = I + f^\lambda$ and $(F^\lambda)^{-1} = I + g^\lambda$ be as in Lemma A.7. Assume that*

$$f^\lambda(0) = 0, \quad |f|_{1,0} \leq \theta/C_N, \quad |f|_{1,s_*} < 1.$$

Let $\rho_1 = (1 - \theta)\rho$ with $1/4 < \rho < 2$ and $0 < \theta < 1/2$. Then

$$\|g\|_{\rho_1;r,s} \leq C_r \{ \|f\|_{\rho;r,s} + \|f\|_{s+1,s} \odot \|f\|_{r,s_*} \}, \\ \|\{u^\lambda \circ (F^\lambda)^{-1}\}\|_{\rho_1;r,s} \leq C_r \{ \|u\|_{\rho;r,s} + |u|_{\rho;1,s_*} \|f\|_{\rho;r,s} \\ + \|u\|_{s+1,s} \odot \|f\|_{r,s_*} + \|u\|_{r,s_*} \odot \|f\|_{s+1,s} + |u|_{\rho;1,0} \|f\|_{s+1,s} \odot \|f\|_{r,s_*} \}.$$

Proposition A.11. *Let r, s, s_* be as in Proposition A.10. Let $F_i^\lambda = I + f_i^\lambda: D_i \rightarrow D_{i+1}$, where D_ℓ are as in Proposition A.8. Assume that $f_i^\lambda(0) = 0$ and $|f_i|_{1,s_*} \leq 1$. Then*

$$\begin{aligned} \|\{u^\lambda \circ F_m^\lambda \circ \cdots \circ F_1^\lambda\}\|_{D_1;r,s} &\leq C_r^m \left\{ \|u\|_{r,s} + \|u\|_{1,0} \sum_{i \leq j} \|f_i\|_{s+1,s} \odot \|f_j\|_{r,s_*} \right. \\ &\quad \left. + \sum_i |u|_{1,s_*} \|f_i\|_{r,s} + \|u\|_{s+1,s} \odot \|f_i\|_{r,s_*} + \|u\|_{r,s_*} \odot \|f_i\|_{s+1,s} \right\}, \end{aligned}$$

where $\|f_i\|_{a,b} = \|f_i\|_{D_i;a,b}$ and $\|u\|_{a,b} = \|u\|_{D_{m+1};a,b}$.

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